the family **MATH.** companion

ARITHMETIC—THE FOUNDATION OF MATH

Ruth C. Sun

The Family MATH Companion

Arithmetic - The Foundation of Math

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Preface

More than six years ago, in response to a need, I started math clubs for students from second grade to eighth grade, teaching arithmetic, pre-algebra, and algebra. It led me to ask and analyze why many students did poorly in math and so few were ready for algebra at eighth grade. The following are some of the findings, I believe, that contribute to the poor performance in math.

* Math study seems to jump from one topic to another with little logical sequence. Students do not know what is essential and what is not.
* Students learn how (the skills), but not what (the concept) and why (the reason) Most of the students do not know how numbers are related.
* Many students learn little at the end of the school year, and lose the "little" they have learned during the long summer vacation.
* When the fall comes, students are back to "square one" and they are tired of going over the same old stuff again.

Failure in math seems to affect a child's self-image more than any other subject. It undermines a child's confidence. It is to correct these problems that "The Family Math Companion," a reference book, was written. The book deals exclusively with three positive numbers and four operations to show "what arithmetic is" and "how it works" in a systematic, logical way. It is intended to lay a solid foundation upon which the students will be able to move to a higher level of math. For that reason, graphing, geometry, etc., are excluded.

The book is intended to serve:

- ***Parents.*** The book is complementary to math books used in school. It is a tool which will give the parents the needed confidence to assist their children in math work at home. We know "failure tends to breed failure; success breeds success." Failure in math is something that can be prevented if the home works together with the school. Mathematics is a unique subject, once a student gets behind, it is very difficult for him or her to catch up.
- ***High school students.*** The book can help high school students who are lost in the maze of math to find a way out. Particularly, it gives an exhaustive treatment of "factors, multiples, and the related concepts and skills" (pp.277-328) which are absolutely essential to do fractions.
- ***College-bound students*** The book provides an excellent review for those who are preparing to pursue a higher level of education.

It is fashionable today to talk about algebra and to make algebra relevant to students. But the question is "How can a student study algebra without the solid arithmetic foundation?" If students are expected to study algebra, then make sure they learn first how to add, subtract, multiply, and divide. First things come first. **Back to the basics!**

I would like to thank the school administrators and teachers of school districts in the Chicagoland area for their assistance in my research. Dr Harry Agabedis and Ms. Anna Horn Kreske deserve special thanks for their support and encouragement and special thanks also to Mrs. June Osborne. Thanks to SSC in South Holland, Illinois for the use of their laser printer and technical assistance from some of their staff Thanks also to Mr. Jerry Spehar for his technical assistance. Finally, thanks to Mr George Stone for some practical suggestions.

Ruth C Sun
April, 1996

How To Use This Book

The Family Math Companion does not contain exercises because it is a **know-how** book. Since it is complementary to the school math books, you can find plenty of exercises in those types of textbooks.

Make Good Use of the Unique Features of This Book:
* Each page illustrates a topic with a step-by-step explanation. Read each word carefully to be sure you understand
* Read those pages as indicated in the cross references. It is the "prior knowledge" you need to understand the topic you are studying.
* The concepts or skills that are easy to get confused are placed side by side for comparison. Study them until you understand the differences.
* Pay attention to the words printed in bold letters. They are key words, or concepts, or rules that you should memorize, once you understand.
* To obtain the most benefit, take time to read the complete "Part" or at least the "Introduction" and the entire "Section" related to the problem you are working By so doing, you will learn mathematics in a systematic way

General Procedure for Using the Book:

1st. Determine the nature of the problem you are working by asking yourself:
  a) Is the problem related to "Numbers & Concepts" or "Operations"?
  b) If operation, to which number is it related - whole numbers? decimals? or Fraction?
  c) Then, which operation - addition? subtraction? multiplication? or division?

2nd. Turn to the "Outline" on page 1 Find the "Part" (Part I, II, etc.) and then the "Section" (A, B, etc.) that the problem is under Take note of the pages given.

3rd Turn to the "Section" as indicated by the pages. Find the "Table of Contents", then going down the list, you will find the topic that is related to the problem you are working on.

Example 1 Subtract 7003 – 129

1st. Determine the nature of the problem. It is a whole number operation subtracting across zero.

2nd Turn to the "Outline" on page 1 Find "Whole Number Operations" (Part II), under that you will find the section "C Subtraction" with the page numbers of 147 - 164.

3rd. Then turn to "Table of Contents" for the subtraction of whole numbers on pages 149 Going down the list you will find the topic "Subtracting Across Zeros" with the page number of 163 Turn to page 163 for the information you need

Example 2. Divide 36 ÷ 1.2.

1st. Determine the nature of the problem. It is a division operation - dividing a whole number by a decimal.

2nd. Turn to the "Outline" on page 1 Find "Decimal Operations" (Part III), under that you will find the section "E. Division" with the page numbers of 245-262.

3rd. Then turn to "Table of Contents" for the division of decimals on page 247 Going down the list you will find the topic "Dividing Whole Numbers By Decimals" with the page number 256. Turn to page 256 for the information you need.

To Do Well in Math Requires:

Learning the mathematical terms and symbols.
Having the prior knowledge of related concepts and skills.
-- A lot of practice.

Of course, you need discipline and concentration Read each word carefully, because mathematical statements are made up of compact and precise sentences.

OUTLINE

Index

"The Family Math Companion" - an educational tool every family should have. It is written to enable the parents to become math tutors at home. The book will lay a solid arithmetic foundation upon which a student will be able to move to a higher level of math.
This is a reference book with the following features:

- *It is user friendly and easy to understand.*
- *It is illustrative. Every page explains mainly one concept or one skill.*
- *It is systematic and logical. It explains why as well as what and how.*
- *It gives extensive cross references to show interconnection of math.*
- *It states the important points repeatedly to call attention of the readers.*

Ruth C. Sun
For many years, had math clubs for students from second grade to eighth grade, teaching arithmetic, pre-algebra, and algebra. She received her M.A. from Wheaton Graduate School and is also the author of *"Personal Bible Study. A How To"* She is married to a scientist.

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Part I. Numbers & Concepts

A. Introduction

Table of Contents

Different Sets of Numbers

A set is a collection of objects, numbers, etc. that go together. The following are different sets of numbers that we use in mathematics:

(1)	**1, 2, 3, ...**	**Natural Numbers**	**Rational Numbers**
(2)	**0, 1, 2, 3, ...**	**Whole Numbers**	
(3)	**... -3, -2, -1, 0, 1, 2, 3, ...**	**Integers**	
(4)	**... -3.5, - 2.25, 0.125 0.33, ...**	**Decimals**	
(5)	**... -1/8, -2/3, 4/1, 38/100, ...**	**Fractions**	
(6)	**π ("pi"), $\sqrt{2}$, $\sqrt{5}$, ...**		**Irrational Numbers**
(7)	**Rational numbers & Irrational numbers**		**Real Numbers**

Note: The three dots means the numbers continue without end.

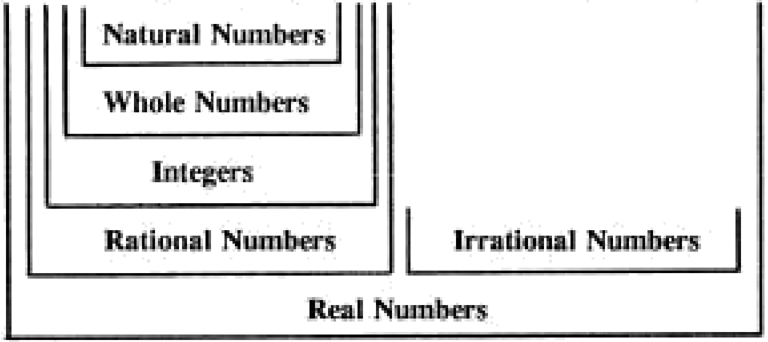
Sets of Numbers And Their Relations

* The set of **natural numbers** (1, 2, 3,...) ***is contained in*** the set of **whole numbers** (0, 1, 2, 3,...). (Natural numbers are also called counting numbers.)

* The set of **whole numbers** (0, 1, 2, 3,...) ***is contained in*** a larger set of **integers** (...,-3, -2, -1, 0, 1, 2, 3,...).

* The set of **integers** ***is contained in yet*** a larger set of **rational numbers** (...,-1/2, -2/5, 3/1,...).

* The set of **rational numbers** and the set of **irrational numbers** ***make up*** the set of **real numbers** as illustrated in the following diagram:



If a square root of a positive integer is not an integer, the square root is irrational

Connection: Knowing the different sets of numbers, their relations and their distinctions, will help you in your study of math later on.

Relations Among Sets Of Positive Numbers

In arithmetic, we deal with only three sets of positive numbers:

(1) **Whole Numbers:** 0, 1, 2, 3, ...
(2) **Decimals:** 0.5, 0.15, 0.33, ...
(3) **Fractions:** 1/2, 1/10, 35/100, ...

Strictly speaking, "0" is neither positive nor negative.

The following show the relations among these three sets of numbers:

1. **All three sets of numbers are part of rational numbers** - a number that can be written in the form a/b (fraction).

2. **Any whole number can be written as a fraction (a/b)** - a knowledge needed to work with fractions **(See p.298)**. But, there is a distinction between "fractions" and "whole numbers" as rational numbers:
   Example: A fraction 1/3 is a fraction, and ***can never be*** a whole number.
   Example: A whole number 7 equals 7/1, a fractional number, ***but*** 7 ***is not*** a fraction.

3. **Any decimal can be written as a fraction.** In fact, decimals and fractions are two different ways of writing the same number **(See p.98)**.

Classification Of Whole Numbers (1) - Even & Odd Numbers

We divide the set of whole numbers into **two major groups** called even and odd numbers. Their differences can be described in the following ways:

* The whole numbers **alternate** between even and odd:

The **even** numbers: 0 2 4 6 8 10 ...
The **odd** numbers: 1 3 5 7 9 ...

The set of whole numbers are ***either*** even ***or*** odd, and **no number is *both*** even and odd. **Zero is considered as an even number.**

* The even numbers can be **divided evenly by 2,** and the odd numbers **can not:**

Even	**Odd**
2 ÷ 2 = 1	3 ÷ 2 = 1 r1
4 ÷ 2 = 2	5 ÷ 2 = 2 r1
6 ÷ 2 = 3	7 ÷ 2 = 3 r1

All even numbers are multiples of 2.

Any number ending in 2, 4, 6, 8, & 0 has 2 as a factor (See p.288).

* The even numbers can always be **grouped by 2,** but the odd numbers always have **1 left over.**

Classification Of Whole Numbers (2) - Prime & Composite Numbers (See p.290).

The set of whole numbers, **except 0 and 1,** can also be classified as prime and composite numbers. For an illustration, let's list the factors of 3, 5, 9, 15, and 21. **Factors are exact divisors** (See p.282).

2 factors	2 factors	3 factors	4 factors	4 factors
3 ÷ 1 = 3	5 ÷ 1 = 5	9 ÷ 1 = 9	15 ÷ 1 = 15	21 ÷ 1 = 21
3 ÷ 3 = 1	5 ÷ 5 = 1	9 ÷ 3 = 3	15 ÷ 3 = 5	21 ÷ 3 = 7
		9 ÷ 9 = 1	15 ÷ 5 = 3	21 ÷ 7 = 3
prime numbers			15 ÷ 15 = 1	21 ÷ 21 = 1

* **Prime Numbers** are the numbers that have *only two factors:* 1 and itself.
* **Composite Numbers** are the numbers that have *more than two factors.*

Prime Numbers vs. Odd Numbers - It is important not to confuse prime numbers with odd numbers. In the above example, 3, 5, 9, 15, 21, are all odd numbers, **but only 3 and 5 are prime numbers.**

Connection: The knowledge of prime and composite numbers is needed in working with fractions.

Cardinal Numbers & Ordinal Numbers

Cardinal Numbers are numbers that are used **to count** or to tell how many.
Example: 1, 2, 5, 8, 21, ...

Ordinal Numbers are numbers that are used to tell, for example,
- **the position in a contest:** first place, second place, etc....
- **the succession of something in a series:** first day of the week, etc....

Writing Ordinal Numbers:

The following shows how to write ordinal numbers:
* First (1st), Second (2nd), Third (3rd), Fourth (4th), Fifth (5th), Sixth (6th), Seventh (7th), Eighth (8th), Ninth (9th), Tenth (10th).
* Eleventh (11th), Twelfth (12th), Thirteenth (13th), Fourteenth (14th), Fifteenth (15th), Sixteenth (16th), Seventeenth (17th), Eighteenth (18th), Nineteenth (19th), Twentieth (20th)
* Twenty-first (21st), Twenty-second (22nd), Twenty-third (23rd), Twenty-fourth (24th),...
* Thirty-first (31st), Thirty-second (32nd) , Thirty-third (33th),...
* Ninety-first (91st),... hundredth (100th).

Connection: We use ordinal numbers in reading exponents **(See p.42)**.

Summary

(Introduction)

- Arithmetic deals only with three positive numbers - whole numbers, decimals, and fractions.
- A rational number is a number that can be written as a quotient of two whole numbers a/b, with 0 excluded as the denominator
- Any whole number can be written as a fraction, but a fraction can not be written as a whole number, except the fractions with 1 as the denominator
- Decimals and fractions are two different ways of writing the same number
- The set of whole numbers can be classified as "even and odd numbers" or as "prime and composite numbers" - with 0 and 1 excluded.

Part I. Numbers & Concepts

B. Whole Numbers

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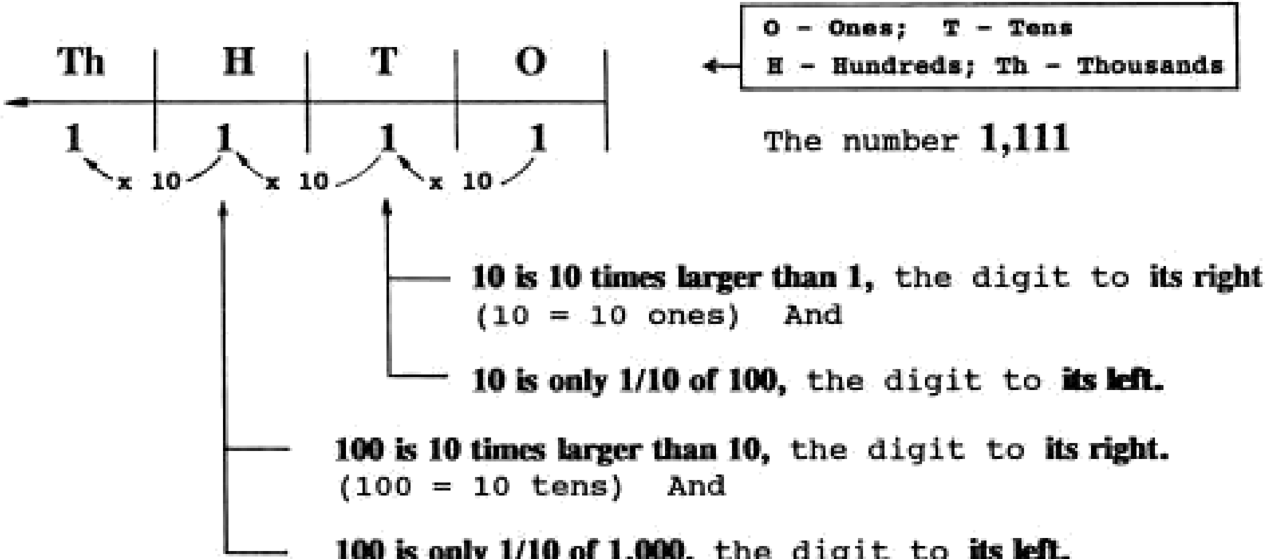
Decimal System Or Base-10 System (See also p.34)

Our number system is called "Decimal System" ("deci" means "ten") or "Base Ten System" because it is **based on 10.** In decimal system, each digit is **10 times larger than the digit to its right** and only **one tenth (1/10) of the digit to its left.** The following shows how decimal system works:



The number **1,111**

10 is 10 times larger than 1, the digit to **its right.**
(10 = 10 ones) And

10 is only 1/10 of 100, the digit to **its left.**

100 is 10 times larger than 10, the digit to **its right.**
(100 = 10 tens) And

100 is only 1/10 of 1,000, the digit to **its left.**

Decimal System - A Place Value System (See also p.109)

The decimal system is also a place value (or positional) system, which means the value of a digit depends on **its place or position in the number.** For example, the number 5,555 has the **face value** of 5, 5, 5, 5, but the **place value** of each "5" is different, depending on its position in the number as seen below:

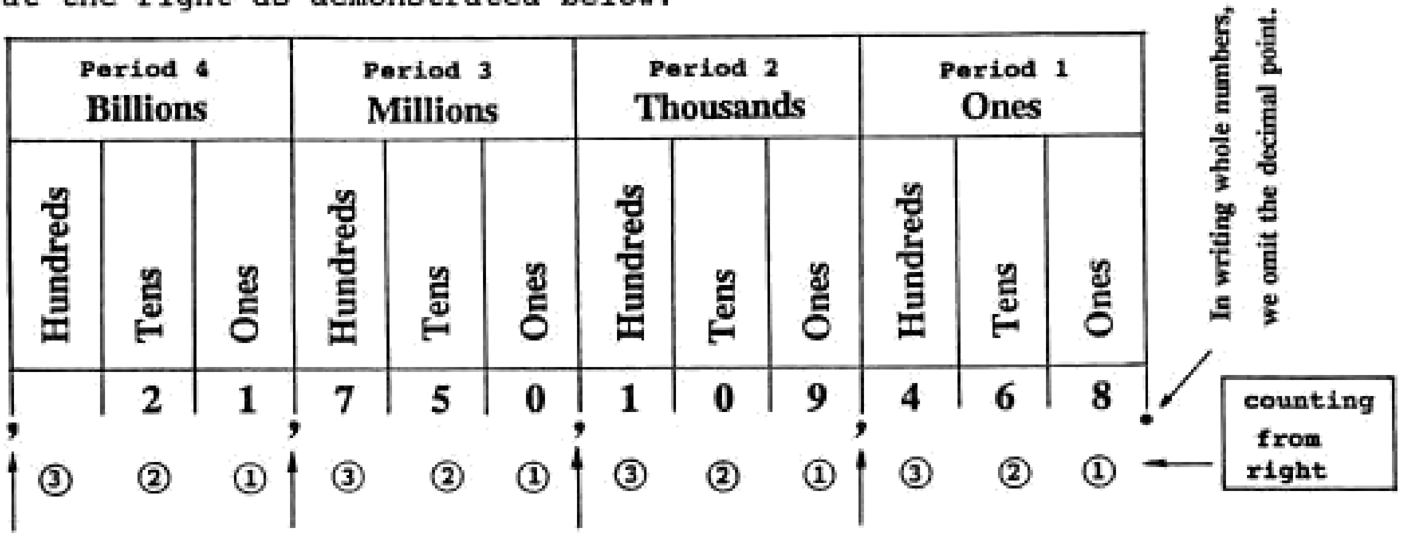
O - Ones; T - Tens
H - Hundreds; Th - Thousands

Th	H	T	O	
5	5	5	5	The number 5,555
			5	the **1st** 5 = **5** (5 **ones**)
		5	**0**	the **2nd** 5 = **50** (**10 times larger** than 1st "5")
	5	**0**	**0**	the **3rd** 5 = **500** (**10 times larger** than 2nd "5")
5	**0**	**0**	**0**	the **4th** 5 = **5000** (**10 times larger** than 3rd "5")

Note: Add 1 zero: 5 becomes **50** (= 5 x **10**), and so on (See p.178).

Chart: Place Value Of Whole Numbers

To make it easier to read and write large numbers, we separate the numbers by **"commas"** into groups of **three** digits called **"periods"** beginning at the right as demonstrated below:



Period 4 Billions			Period 3 Millions			Period 2 Thousands			Period 1 Ones		
Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones	Hundreds	Tens	Ones
	2	1	7	5	0	1	0	9	4	6	8
③	②	①	③	②	①	③	②	①	③	②	①

In writing whole numbers, we omit the decimal point.

counting from right

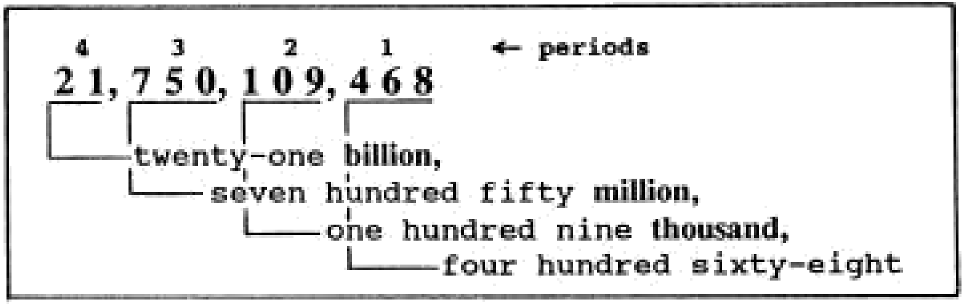
Study the chart until it is firmly fixed in your mind. You need the knowledge in reading and writing large numbers and in arithmetic operations.

Translating Numbers In Numerical Form Into Words

Example: Write 21,750,109,468 in word form.

Step 1. Determine the **largest period** of the number.

The number is in the billions. (4th period)



Step 2. **Start from left,** read or write each group of **three digits** in the normal way followed by the **period name** and **a comma.** Repeat this until you reach the ones period. The word name for 21,750,109,468 is

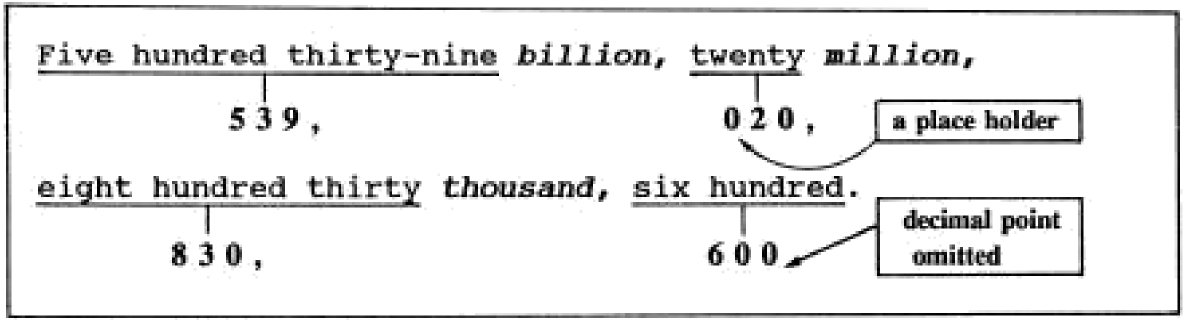
"twenty-one *billion*, seven hundred fifty *million*, one hundred nine *thousand*, four hundred sixty-eight".

Translating Numbers In Words Into Numerical (Standard) Form (1) (Review p.18)

Example: Write "five hundred thirty-nine billion, twenty million, eight hundred thirty thousand, six hundred" in numerical form.

Step 1. Remember that numbers are **grouped in three digits with a comma.**

Step 2. Read the number and translate the word name into digits and **place a "comma" where the period name occurs** (billion, million, thousand). **Use zeros as place holders in all vacant places.**



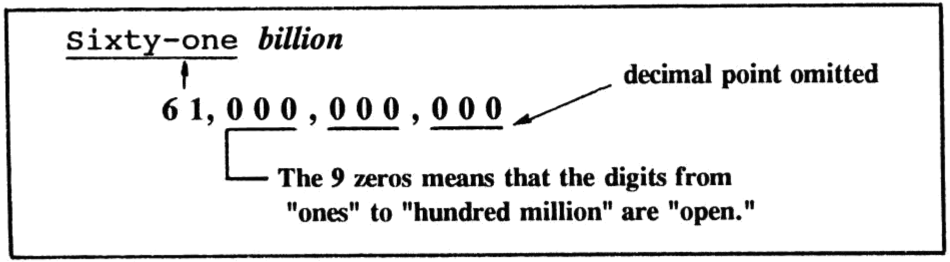
The numerical form of the number is **539,020,830,600.**

Translating Numbers In Words Into Numerical (Standard) Form (2)

Example: Write "sixty-one billion" in numerical (or standard) form.

Step 1. Remember that numbers are **grouped in three digits with a comma.**

Step 2. **Use "0" as a "place holder"** so that the digit "6" will be in the "billions" place and not in "ones" or other place.



The numerical form of the number is **61,000,000,000**

Remember: one thousand - 1,000 - 3 zeros
one million - 1,000,000 - 6 zeros
one billion - 1,000,000,000 - 9 zeros

Things To Remember In Translating Numbers

To translate numbers in numerical form into words:

* The name of the first period "ones" is not named.
* Write the period names in singular (**without "s"**) instead of plural.
* The word **"and"** is **never used** in naming/reading **whole numbers.**
* Use a "hyphen" for two-digit numbers between 21 and 99, except 30, 40, 50, 60, 70, 80, 90.

To translate numbers in words into numerical form:

* **Each period must have 3 places, so use zero(s) as place holder(s), if necessary.**
* The comma is optional for 4-digit numbers. For example, 5000 and 5,000 are both acceptable.

To read 4-digit numbers:

* Translate the 4-digit numbers into the fewest possible words. For example, 1503 can be read "fifteen hundred three," instead of "one thousand five hundred three."

Connection: Every whole number has a decimal point at the end of ones (units) digit, which is omitted. You need to know this in writing "scientific notation," doing "long division," writing "money," etc.

Mastering The Concept Of Place Value & Decimal System

If you want to do well in Math, you need to understand our **number system - the decimal system** and **the place value system.** For example, If you know the concept of **place value** well, you will be able to do the following works with little difficulty:

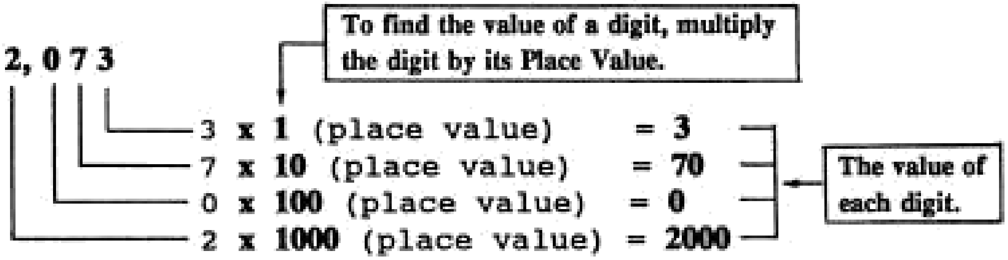
- Reading & writing large numbers
- Writing numbers in expanded form
- Writing numbers in standard form
- Comparing & Ordering
- Rounding
- Estimating
- Adding
- Subtracting
- Multiplying
- Dividing
- Regrouping - Carrying & Borrowing
- Decimals & Fractions
- Money, and so on.

And, if you understand **decimal system** well, you will find it easy to work with powers of 10, multiples of 10, scientific notation, etc.

Writing Numbers In Expanded Form (Review first "Place Value Chart" p.18)

To write a number in expanded form is to write the number as the **sum (addition) of the value of each digit.** The following shows how to write 2073 in expanded form.

Step 1. Find the value of each digit.



2, 0 7 3

To find the value of a digit, multiply the digit by its Place Value.

3 x **1** (place value) = **3**

7 x **10** (place value) = **70**

0 x **100** (place value) = **0**

2 x **1000** (place value) = **2000**

The value of each digit.

Step 2. Write the number as the **sum** of the value of each digit.

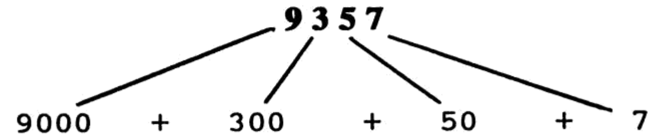
standard numeral → **2,073 = 2,000 + 70 + 3** ← expanded form

The zero (0) digit in standard form is omitted in expanded form.

Writing Standard Numbers In Exponential Notation (Read pp.36,45)

We can write numbers in expanded form or in exponential notation because **the place value of each digit is a power of 10** (See "Decimal System" p.16) To write in exponential notation, first write the number in expanded form and then in exponents. The following shows how to write 9357 in exponential notation:

Write the number 9375 in expanded form:



As the sum of the values of the digits.

$$9(1000) + 3(100) + 5(10) + 7(1)$$

As the sum of the products of each digit and its place value (powers of 10).

$$9(10^3) + 3(10^2) + 5(10^1) + 7(10^0)$$

Then write the powers of 10 in exponent form.

Note: If you know the **place value** and **powers of 10**, you can write directly.

Writing Numbers in Expanded Form as Standard Form (1)

There are two ways to write 800,000 + 50,000 + 6,000 + 300 + 1 in standard form.

Method 1. Column Addition

First, line up the numbers vertically according to their place value beginning from right, from the smallest number to the largest number. Then, add. **The sum is the standard numeral.**

	place value
1	ones
00	tens
300	hundreds
6,000	thousands
50,000	ten-thousands
+ 800,000	hundred-thousands
856,301	

Standard numeral

According to the decimal system, each number should have one more zero (10 times larger) than the number right above it.

The place for the "tens" digit is listed above, on purpose, to show that "zero digit" (no tens) is omitted in the expanded form, but it is included in the standard numeral as a place holder.

Writing Numbers in Expanded Form as Standard Forms (2)

Use the following method to write 800,000 + 50,000 + 6,000 + 300 + 1 in standard form if you know the place value well.

Method 2. Direct Method

1st. Start from left to right, from the largest number to the smallest, write the "nonzero-digits" (8, 5, 6, 3, 1) together in decreasing order according to their place values. Use "0" for "vacant" digit if necessary. Remember **each period contains three digits.**

2nd. Check to see if the standard numeral has as many digits as there are in the largest-place-value-digit (the first number from the left).

largest-place-value-digit: **800,000** ← 6 digits
standard numeral: **856,301** ← 6 digits

Zero (0) holds the "tens" place so that the other digits, 3, 6, 5, 8 can be in their proper places.

Connection: Writing numbers in expanded form for arithmetic operations can help beginners to understand how operations work involving regrouping such as carrying and borrowing. (See pp.139, 159)

Symbols for Comparing Numbers (Including Decimals, Fractions): > & <

Two of the symbols used in comparing and ordering numbers are > and < as seen in the following examples:

	Compare:			**Read:**
(lesser number)	$3 < 4$	(larger number)	--	3 **is less than** 4.
(larger number)	$4 > 3$	(lesser number)	--	4 **is greater than** 3.

< means **"is less than"**   > means **"is greater than"**

Remember:

* The symbol **is always pointed to the lesser** of the two numbers.

* In reading comparison, always read from **left to right.**

* In ordering numbers, **the symbols must point in the same direction.**

Other Symbols used in comparing numbers are:

= means "is equal to"

≠ means "is not equal to"

≥ means "is equal to or greater than"

≤ means "is equal to or less than"

Comparing and Ordering Whole Numbers (Review first "Place Value" p.18)

The easiest way is to line up the numbers according to their place values. Then **compare digits of the same place value beginning from the left.**

a) **Comparing Two Numbers:** 18,354 & 15,989

18,354

15,989

equal 8 > 5

Begin from left, **find the first pair of digits that are not equal.** Since 8 > 5, 18,354 > 15,989 or 15,989 < 18,354

b) **Comparing & Ordering More Than Two Numbers:** 37,006, 6,792, 31,885, 39,954

General rule for whole numbers: *The fewer* the digits, *the smaller* the number; *the more* the digits, *the greater* the number.

equal 1 < 7 and 7 < 9

37,006
6,792
31,885
39,954

Compare two numbers at a time **first, then order** them from least to greatest:

6,792 < 31,885 < 37.006 < 39,954

(The symbols must point in the same direction.)

Note: To order from largest to least, just **reverse the order.**

Rounding Off (Review first "Place Value Chart" p.18)

Rounding off a number is to express the number as an **approximation** to a required place (nearest ten, hundred, etc.), the place depending on the **degree of precision desired.** The following shows how to round 7548 and 9152 to the nearest hundred, respectively.

(a) $7\underline{5}48$

(b) $9\underline{1}52$

1. **Mark off the digit** to which the number is to be rounded with an underline or any mark of your choice.

Underline the digit "hundred" (the 3rd digit from the right).

4 is less than 5, so the digits to its right, 48, is rounded down to 00. The rounded number, 7500, is smaller than the actual number.

(a) $7\underline{5}48$

$$\begin{array}{cccc} 7 & \underline{5} & 4 & 8 \\ & & \downarrow & \downarrow \\ 7 & 5 & 0 & 0 \end{array}$$

2. If the digit *to its right* is **less than 5** (4, 3, 2, 1), replace all the digits to its right with zeros.

7548 is rounded to the nearest hundred.

**5 or greater than 5 (> 5), so 52 was rounded up to 100. ("+1" = + 100)
The rounded number, 9200, is larger than the actual number.**

$$\overset{+1}{}$$
(b) $9\,\underline{1}\,5\,2$

$$\downarrow\ \downarrow$$

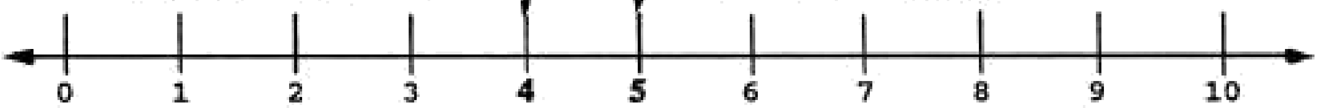
$9\,2\,0\,0$

3. If the digit *to its right* is **5 or greater than 5** (5, 6, 7, 8, 9), **add 1** to the underlined digit and replace all the digits to its right with zeros.

Demonstrating The Rules On A Number Line:

4 is closer to 0 than to 10.

5 is closer to 10 than to 0.



If **4** or less (< 5)
Rounding **down** to 0

If **5** or larger than 5 (> 5)
Rounding **up** to 10 (+ 1)

The rounded number will be less than the actual number.

The rounded number will be larger than the actual number.

Rounding Up or Rounding Down

Study the following examples carefully and note the result of rounding a number up or down. Will it affect estimation? Think!

Example: Round 2,499 and 1,500 to the nearest thousand, respectively.

$$\begin{array}{l} \underline{2},\overset{\downarrow}{4}\,9\,9 \\ \quad\ \downarrow\ \downarrow\ \downarrow \\ \underline{2},0\,0\,0 \end{array}$$

Rounding Down:

Since 4 < 5, 499 was rounded down to 000.
The rounded number is **less than** the actual number.

When rounding to the nearest thousand, 2,499 is the largest number and 1,500 is the least numbers that are rounded to 2,000.

$$\begin{array}{l} \overset{+1}{\underline{1}},\overset{\downarrow}{5}\,0\,0 \\ \quad\ \downarrow\ \downarrow\ \downarrow \\ \underline{2},0\,0\,0 \end{array}$$

Rounding Up:

Since 5 = 5, 500 was rounded up to 1000: +1.
The rounded number is **more than** the actual number.

<u>Connection:</u> Learn the skill of rounding, it is one of the methods used in **estimating** sums, differences, products, quotients, etc.

Rounding to various places - A number can be rounded to different places depending on how close to the actual value your estimate needed to be. For example, let's round 62,594 to:

(a) the nearest **ten thousand:**

$$\begin{array}{ccccc} & \downarrow & & & \\ \underline{6} & 2, & 5 & 9 & 4 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 0, & 0 & 0 & 0 \end{array}$$

(b) the nearest **thousand:**

$$\begin{array}{ccccc} +1 & & \downarrow & & \\ 6 & \underline{2}, & 5 & 9 & 4 \\ & & \downarrow & \downarrow & \downarrow \\ 6 & 3, & 0 & 0 & 0 \end{array}$$

When a number is rounded to the **largest** or **larger place,** it gives a **"rough"** estimate. The rounded number will be far over or under the actual value depending on whether the number is rounded up or rounded down. (a) & (b)

(c) the nearest **hundred:**

$$\begin{array}{ccccc} & & +1 & \downarrow & \\ 6 & 2, & \underline{5} & 9 & 4 \\ & & & \downarrow & \downarrow \\ 6 & 2, & 6 & 0 & 0 \end{array}$$

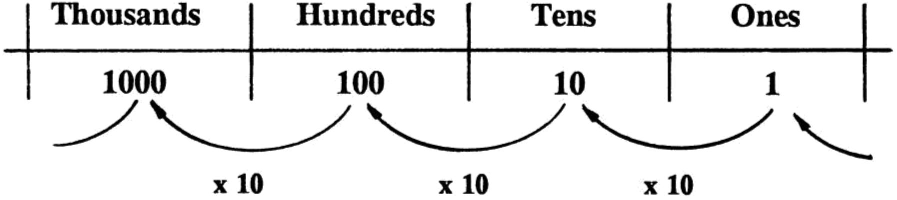
(d) the nearest **ten:**

$$\begin{array}{ccccc} & & & & \downarrow \\ 6 & 2, & 5 & \underline{9} & 4 \\ & & & & \downarrow \\ 6 & 2, & 5 & 9 & 0 \end{array}$$

When a number is rounded to a **lesser place,** the rounded number will be **closer** to the actual value whether the number is rounded up or rounded down. (c) & (d)

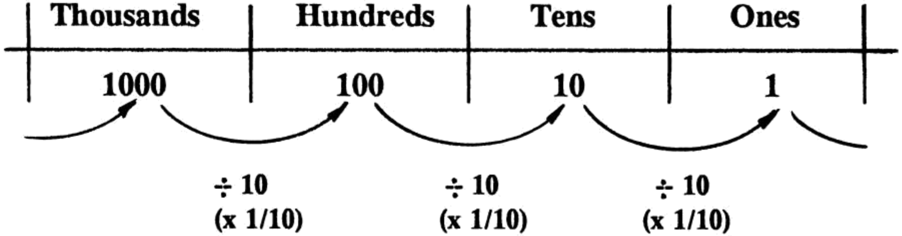
Another Look At The Decimal System (See also p.16)

The following gives you another look at how our decimal system works. It can be extended in both directions without end. Moving ***to the left,*** the numbers become **larger and larger.** Moving ***to the right,*** the numbers become **smaller and smaller** and eventually **leads to decimals** - numbers less than 1 in value.



Thousands	Hundreds	Tens	Ones
1000	100	10	1

Add 1 zero when **multiplying by 10.**



Thousands	Hundreds	Tens	Ones
1000	100	10	1

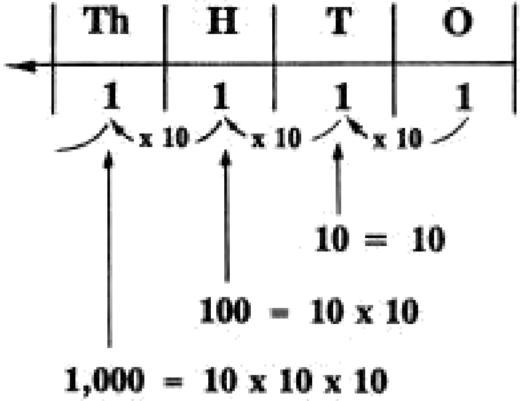
Drop 1 zero when **dividing by 10.**

To divide by 10 is the same as to multiply by its reciprocal, 1/10.

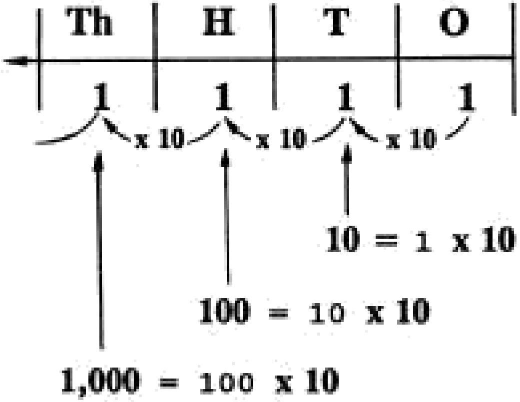
Decimal System, Powers Of 10, & Multiples Of 10

Since our number system is based on 10, the **value of each place** can be expressed in "powers of 10" (multiplication of 10s) or in "multiples of 10" (10 multiplied by any number) as shown below:

In Powers of 10:



In Multiples of 10:



Note: Study carefully the charts found on the next two pages. You will find them useful in working with **powers of 10 (exponents)** and **multiples of 10**, etc.

Chart: Expressing Place Value In Various Forms

The following two tables show that **the place value of our decimal system** can be written in various forms.

In Word Name	In Standard Form	In Multiples of 10 (x 10)	In Exponents
ones	1	= 1	= 10^{0}
tens	10	= 1 **x 10**	= 10^{1}
hundreds	100	= 10 **x 10**	= 10^{2}
one thousands	1,000	= 100 **x 10**	= 10^{3}
ten thousands	10,000	= 1,000 **x 10**	= 10^{4}
hundred thousands	100,000	= 10,000 **x 10**	= 10^{5}
one millions	1,000,000	= 100,000 **x 10**	= 10^{6}
ten millions	10,000,000	= 1,000,000 **x 10**	= 10^{7}
hundred millions	100,000,000	= 10,000,000 **x 10**	= 10^{8}
one billions	1,000,000,000	= 100,000,000 **x 10**	= 10^{9}
ten billions	10,000,000,000	= 1,000,000,000 **x 10**	= 10^{10}
hundred billions	100,000,000,000	= 10,000,000,000 **x 10**	= 10^{11}

Note: The number of zeros in Standard Form = the exponent number.

Chart: Expressing Place Value In Various Forms (See also pp.44,45)

The powers of 10, the basis of our decimal system, can be expressed in "exponent form," "factor form," and "standard form." (See also p.42)

For zero power & first power see p.45

Exponent Form	Factor Form	
10^0	= 1	
10^1	= 10	
10^2	= 10 x 10	
10^3	= 10 x 10 x 10	*(Thousands: 3 zeros.)*
10^4	= 10 x 10 x 10 x 10	
10^5	= 10 x 10 x 10 x 10 x 10	
10^6	= 10 x 10 x 10 x 10 x 10 x 10	*(Millions: 6 zeros)*
10^7	= 10 x 10 x 10 x 10 x 10 x 10 x 10	
10^8	= 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10	
10^9	= 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10	*(Billions: 9 zeros)*
10^{10}	= 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10	
10^{11}	= 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10 x 10	

Note: The exponent number = the number of factors.

Multiplication & Exponents

Exponents are the **shorthand way** of writing multiplication when **one factor is used a number of times.** For example, instead of writing 8 x 8 x 8 x 8 x 8, we can simply write 8^5 in exponential notation. Exponents are multiplication, but with this difference:

In Multiplication:

Factors are ***different*** →

$$\underbrace{2 \times 8 \times 5}_{\textbf{factors}} = \underset{\substack{\uparrow \\ \textit{product}}}{80}$$

In Exponents:

Factors are ***identical*** →

$$\underbrace{6 \times 6 \times 6}_{\textbf{factors}} = 6^{\underset{\substack{\uparrow \\ \textit{power}}}{3}} = 216$$

<u>Connection:</u> Exponents are used widely in math because it is a convenient way of writing large numbers.

Exponents & Powers Of 10

Powers of 10 are exponents. The difference between the two is:

In Exponents:

$9 \times 9 \times 9 \times 9 = 9^4$

$y \cdot y \cdot y \cdot y \cdot y \cdot y = y^6$

In exponents, the "***base***" can be ***any number*** (excluding zero) or ***variables.***

Variables are letters, such as a, b, c, x, y, z, etc., that stand for numbers.
A "centered dot" or parentheses (y)(y) can be used to indicate multiplication. (See p.113)

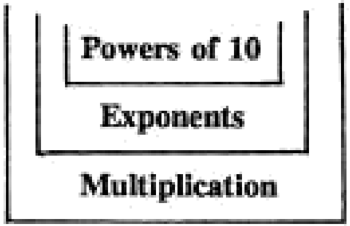
In Powers of 10:

$10 \times 10 \times 10 = 10^3$

In powers of 10, the "***base***" is ***always 10.***

Connection: The powers of 10 in exponents are used in writing scientific notation for very large and very small numbers. (See p.74)

Relating Multiplication, Exponents, & Powers Of 10



Points To Remember:

Multiplication, exponents, and powers of 10 are related as you see in the diagram on the left. We may consider that:

* The power of 10 is **a special case** of the exponent.

* And the exponent is **a special case** of multiplication.

Connections:

Exponents are very useful. For example, we can write:
- **place value** in exponents (powers of 10). **(See pp.36, 72)**
- numbers in **expanded form** with exponents (powers of 10). **(See p.25)**
- **prime factorization** with exponents. **(See p.319)**
- very large/small numbers in **scientific notation** (powers of 10). **(See p.74)**

And exponents are used extensively in algebra.

Exponents - Raising A Number To Powers (see p.38)

When a number is multiplied by itself once or more times, the number is raised to a power. Let's take the number 5 as an example:

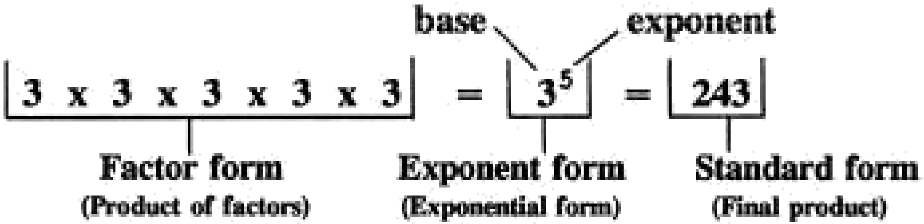
5 multiplied by itself:	5 is raised to the:	The exponents is read:
once (5 x 5)	2nd power (5^2)	"five to the **second power**" or "five **squared**"
twice (5 x 5 x 5)	3rd power (5^3)	"five to the **third power**" or "five **cubed**"
three times (5 x 5 x 5 x 5)	4th power (5^4)	"five to the **fourth power**" ...

Note: Only the "second" and the "three" powers have different names.
"**Square numbers**" are **two-dimensional** numbers.
"**Cubic numbers**" are **three-dimensional** numbers.

Reading and Writing Exponents

Base - The number (3) which is used as factors repeatedly is called base.

Exponent - The raised small number is called exponent. It tells how many times the base (3) is used as factors.



Reading Exponents - It is important that you are able to read exponential notation correctly. For example, to read 3^5

- First, read the base (3): **"three"**
- Then, add the words "to the": **"three to the"**
- Finally, read the exponent (5) as an ordinal number with the word "power": **"three to the fifth power."** 3^5

Ordinal numbers are numbers that show order or position: 1st, 2nd, 3rd,... (See p.11)

Example: Write 7 x 7 x 7 x 7 x 7 in exponent form.

Step 2. Count the **number of factors** (5).

① ② ③ ④ ⑤

$$7 \times 7 \times 7 \times 7 \times 7 = 7^6$$

Step 3. Write the **exponent: 5.**

The number of times the base has been used as a factor.

Step 1. Write the **base: 7.**

The number (7) used as a factor repeatedly.

Example: Write 3^4 (a) in factor form, (b) in standard form.

The exponent "4" means that there are 4 factors.

① ② ③ ④

(a) $3^4 = 3 \times 3 \times 3 \times 3$

(b) $3^4 = 81$ (final product)

Remember: 3^4 does not mean 3 x 4.

Working With Powers Of 10 (Read "Rule for Multiplying by 10, etc.," p.178)

If you want to get the final product of 5^6 or 2^9 , you have to multiply out the long way, two factors at a time unless you use a calculator. But to find the final product of powers of 10 can be done **with no work.** Here is the shortcut.

Powers of 10	**Number**	**Shortcut**
10^8	100,000,000	Write 1, followed by **exactly** the same number of zeros as the exponent number.
10^7	10,000,000	
10^6	1,000,000	
10^5	100,000	

Reason behind the shortcut:

$\mathbf{10^3} = 10 \times 10 \times 10$	The exponent (3) indicates the number of times 10 multiplied by itself or the number of factors 10.
$10 \times 10 \times 10 = \mathbf{1,000}$	The rule is to add one zero to the number each time it is multiplied by 10 (or add one zero for each factor 10).

The shortcut utilized the rule for multiplying by a power of 10.

The shortcut also works when writing a number as a power of 10 in exponent (or exponential form). It is just the other way around.

Number	Powers of 10
100,000	10^5
10,000	10^4
1,000	10^3
100	10^2

Shortcut

Write 10 (the base), then count **the number of zeros, NOT the number of digits.** Write that total as the exponent number.

First Power & Zero Power - If the numbers above are continued in decreasing order, we would notice the following pattern:

Number	Exponents
10,000	10^4
1,000	10^3
100	10^2
10	10^1
1	10^0

Since each successive number decreases by a factor of 10, we can conclude that:

10^1 - **10 to the 1st power is 10.**

10^0 - **10 to the zero power is 1.**

Remember: Any non-zero number to the 1st power is the number Any non-zero number to the zero power is 1.

Summary
(Whole Numbers)

* Our number system is called the decimal system or the base-10 system because each place (digit) is 10 times larger than the place (digit) to its right.
* The decimal system implies a place value system which means the value of a digit depends on its place in the number.
* To make it easy to read, the whole numbers are grouped into periods of three digits. Any whole number can be written in standard numeral, in word, or in expanded form.
* To compare and order the numbers, line up the numbers according to their place values, then compare the digits of same value beginning from the left. The larger the number, the greater the value.
* Rounding gives an approximation when the exact answer is not required. The rounded-up number is always larger than the actual number
* Since our number system is a decimal system, the value of each place can be expressed by the powers of 10 (the multiplication of 10s) The powers of 10 are very easy to work with.
* Exponents are the shorthand way of writing numbers when one factor is used repeatedly

Part I. Numbers & Concepts

C. Decimals

Table of Contents

Decimal Point - The Dividing Point (Review first "Decimal System" p.34)

Our decimal system works also to the right of ones digit. We use **the decimal point** *to divide* **the whole numbers part from the decimals** as seen below:

Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths
100	**10**	**1**	•	**0.1**	**0.01**	**0.001**

To the left are: ← **decimal point** → *To the right* are:

- * *Whole numbers*
- * Numbers equal to or greater than 1 (≥ 1)

- * *Decimals*
- * Numbers less than 1 (< 1)

value increases ← **moving to the left**

moving to the right → **value decreases**

This is how the decimal system works.

Decimals & Fractions (Read also p.225)

Decimals and fractions are **two different ways** of writing the same number. If we write 0.524 in expanded form, you will notice that each decimal digit can be expressed in both decimal and fraction.

.524   (.524 = 524/1000)

In decimal	**.5**	+	**.02**	+	**.004**
	5 tenths		**2 hundredths**		**4 thousandths**
In fraction	$\frac{5}{10}$	+	$\frac{2}{100}$	+	$\frac{4}{1000}$
	(1/2)		(1/50)		(1/250)

Reducing to lowest terms changes a decimal fraction to a common fraction.

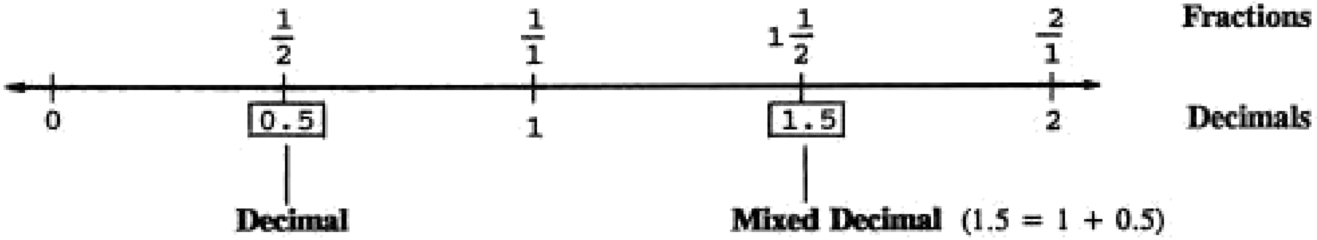
Connection: If you know that decimals can be written in fractions or the other way around, it can help to simplify the computation.

Example: .25 x 80 = 1/4 x 80 = 20

Relating Whole Numbers & Decimals/Mixed Decimals (see also p.93)

On a number line, you can find **all decimals and mixed decimals. They are numbers that lie between some pairs of whole numbers.** The difference between the two is:

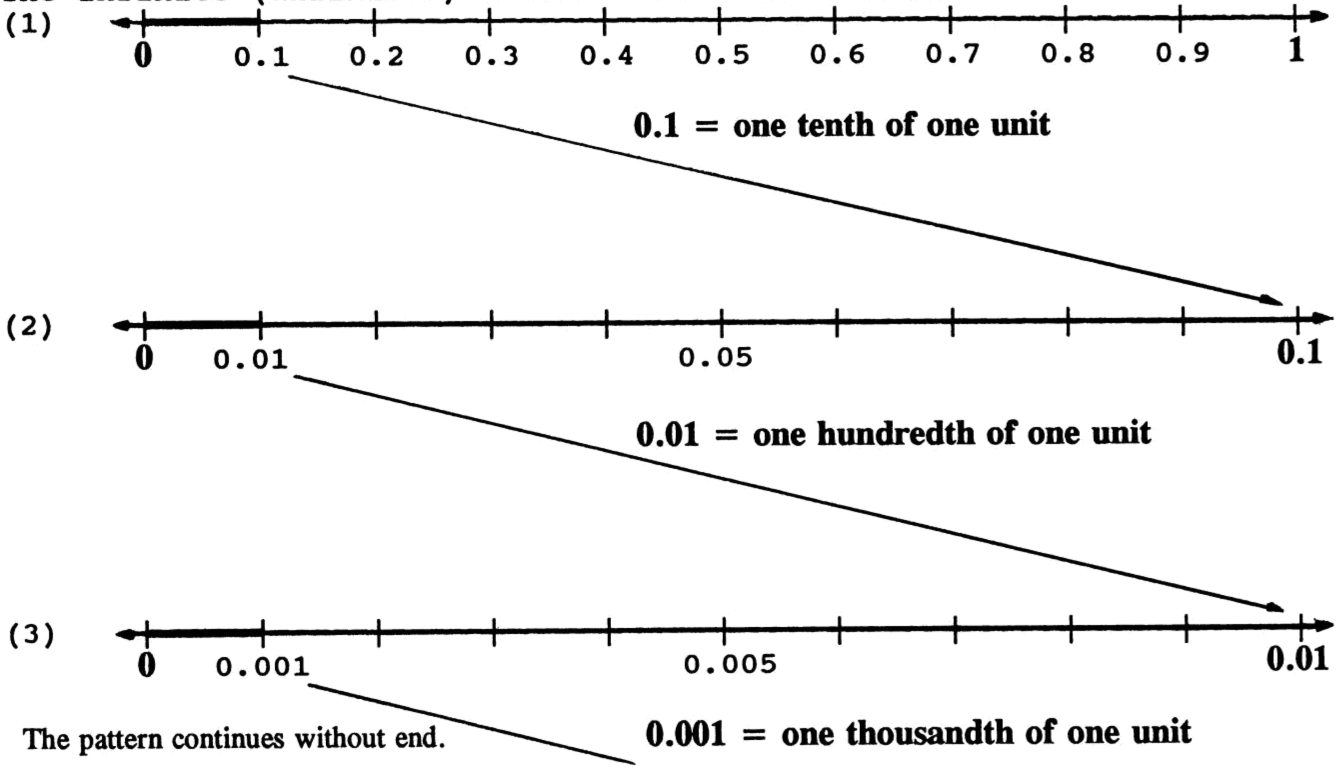
* **Decimals** are *less than 1* (< 1) - (no whole number part)
* **Mixed decimals** are *greater than 1* (> 1) - (whole number + decimal)



With the help of number lines, we can see a **major difference** between ***the set of whole numbers*** and ***the set of decimals/fractions***:

* Between any pair of whole numbers, 0 & 1, 1 & 2, 5 & 6, 11 & 12, etc., **it is impossible to find another whole number.** However,
* Between any pair of whole numbers or any two decimals (fractions), there are an **infinite (unlimited) number** of decimals/fractions. On the next page you will see unlimited decimals just between 0 & 0.1:

The infinite (unlimited) decimals between 0 & 0.1:



0.1 = one tenth of one unit

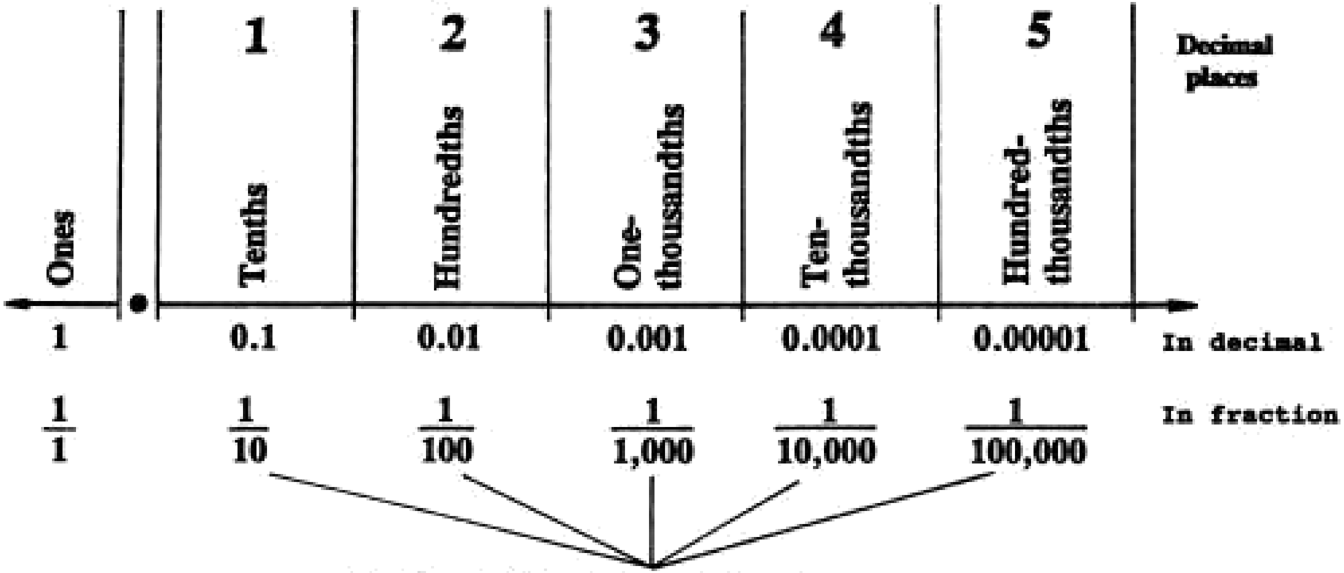
0.01 = one hundredth of one unit

0.001 = one thousandth of one unit

The pattern continues without end.

Chart: Place Value Of Decimals (Review first p.50)

The chart below gives the place value of the digits ***to the right* of the decimal point.** On a number line, you will find these decimal points lie between 0 and 1 as illustrated on the previous page.

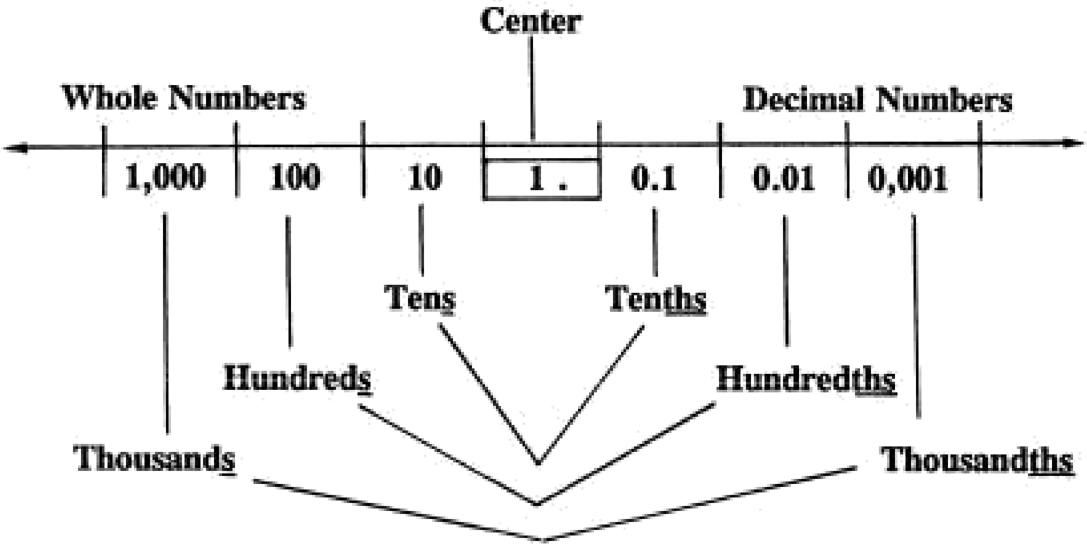


		1	2	3	4	5	Decimal places
Ones	●	Tenths	Hundredths	One-thousandths	Ten-thousandths	Hundred-thousandths	
1		0.1	0.01	0.001	0.0001	0.00001	In decimal
$\frac{1}{1}$		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$	$\frac{1}{10,000}$	$\frac{1}{100,000}$	In fraction

The denominators are powers of 10

The following suggestion may help you to visualize the place value of whole numbers and decimal numbers.

Consider 1 and decimal point (.) as the center.



Do you see the relation?

Number of Decimal Places Or Decimal Digits (Review first p.54)

You need to have the concept of "decimal places" and their names before you can translate correctly a decimal in words (such as "three hundredths") into a decimal in standard numeral or vice versa. **The number of decimal places** means **the number of digits** a number has **after the decimal point.** Compare with "Digits" on page 225.

Place after decimal point	Word name of the place	Number of "decimal places"
1st	Tenths	1 place
2nd	Hundredths	2 places
3rd	Thousandths	3 places
4th	Ten thousandths	4 places
5th	Hundred thousandths	5 places

Memorize the first four place names and the number of decimal places!

Points To Remember:

* The "**th**" at the end of word name indicates a decimal number.

* The "**and**" and "**th**" at the end of word name indicate a mixed number.

Translating Words into Decimals and Fractions (Review first p.54)

Decide first ***how many decimal places*** the number has as indicated by the ***word name of the place,*** then you know whether or not zero(s) are needed to hold the vacant place(s). Study carefully the following examples.

1. Write "six and four thousandths" in decimal and fraction.

six and four *thousandths* = **6.004** or **6 4/1000**

6 . 004

The word name "**thousandths**" indicates the number has **3 decimal places.** Since 4 is in "thousandths" place, "**0s**" are needed ***to hold*** the ***"tenths"*** & ***"hundredths"*** places.

2. Write "nine tenths" in standard form or standard numeral.

nine *tenths* = **0.9** or **9/10**

0 . 9

Write a zero in "**ones**" place for numbers **less than 1**, because it is easy to mistake **.9** (nine tenths) for **9** (nine).

Reading & Writing Decimals in Words - Use "Place Value Charts" for help.

1. Reading **Decimals** - numbers between 0 and 1.

(a) $0.\underline{409}$

four hundred nine thousandths

Read the decimal as you would with whole number Then, read the place-value name of the last digit (3rd place).

2. Reading **Mixed Decimals** - numbers greater than 1.

(b) $2.\underline{016}$

two and sixteen thousandths

Read the whole number part first, then "and" for decimal point, then the fractional part (decimal).

3. Reading **Equivalent Decimals** - decimals which have same value.

(c) $0.\underline{7} = 0.\underline{70}$

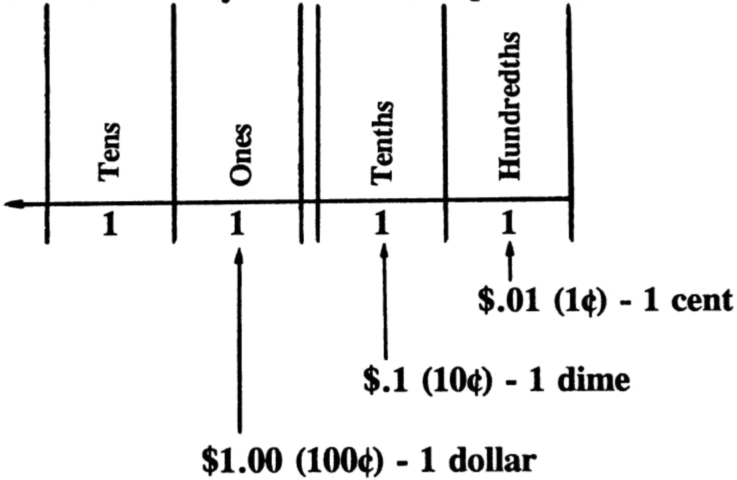
seven *tenths* seventy *hundredths*

Place zero(s) after a decimal does not affect its value, but it does affect the way the decimal is read.

Connection: We use "equivalent decimals" in **comparing & ordering** decimals.

Decimals and Money (Read also p.227)

Our money system works in the same way as our decimal system except it involves only two decimal places - the "tenths" (dimes) & "hundredths" (cents).



1 dollar = 2 half dollars
= 4 quarters
= 10 dimes
= 20 nickels
= 100 pennies

1 quarter = 2 dimes & 1 nickel
= 5 nickels
= 25 pennies

1 dime = 2 nickels
= 10 pennies

1 nickel = 5 pennies

There are two ways to write money:

- using dollar sign ($) and decimal point (.) → $.00
- using the cent sign (¢) for the amount less than one dollar.

Reading and Writing Money (Review first p.59)

Read and write money the way you would with whole numbers and decimals; however, add the word "dollar" after the whole numbers and "cent" after the decimal numbers.

(a) **$26.10**

→ **twenty-six dollars**

→ **and**

→ **ten cents**

Reading money:
1st, read the whole number part **ending with "dollar"**. Then, read **the decimal point with "and"**. Finally, read the **decimal part** followed by the word **"cent"**.
Read: "twenty-six dollars and ten cents."

(b) **"five dollars and fifty-nine cents"**

$5.59

Writing money:
1st, write the dollar sign "**$**" followed by the whole number. Then write the decimal point "**.**" for "and", followed by **2 decimal numbers** (2 places).

(c) **"forty-seven cents"**

$.47 or 47¢

Write the dollar sign with a dot "**$.**" followed by the decimal numbers. Or write the decimal numbers with **cent sign.**

Rounding Decimals & Money (Review first "Rounding Off," p.30)

Rounding decimals and rounding money follow the same rule as rounding whole numbers. The digit to the right of the place you are rounding determine whether to round up or to round down. The following examples show the one difference between rounding money and rounding decimal.

Example 1: Rounding $19.58 to the nearest dime.

$$\$19.\overset{\downarrow}{5}8 \longrightarrow \$19.60$$

With money, we always keep **two** decimal places - the "*dime*" (tenths) and the "*cent*" (hundredths). (See also p.59)

Example 2: Rounding 19.58 to the nearest tenths.

$$19.\overset{\downarrow}{5}8 \longrightarrow 19.6$$

With decimal, we drop zero(s) after the decimal number. **Dropping** the "**0s**" after a decimal ***does not*** change its value: 19.60 = 19.6. They are equivalent decimals. (See also p.67)

Writing Decimals & Money In Expanded Form (Review first, p.24)

The process of writing decimals and money in expanded form is the same as writing whole numbers in expanded form. **The key is to recognize the place value of each digit.** The following examples show how to write (a) 32.759 and (b) $164.36 in expanded form:

(a) **32.759**

$= 3(10) + 2(1) + 7(0.1) + 5(0.01) + 9(0.001)$ **digit x (place value)**

$= \mathbf{30 + 2 + 0.7 + 0.05 + 0.009}$ ← **32.759 in expanded form**

(b) **$164.36**

= \$1(100) + \$6(10) + \$4(1) + \$3(0.1) + \$6(0.01)   **digit x (place value)**

= **\$100.00 + \$60.00 + \$4.00 + \$.30 + \$.06** ← **\$164.36 in expanded form**

To avoid errors, always write money with two decimal places: $100.00 instead of $100, and $.30 instead of $.3

Writing Decimals (Expanded Form) in Standard Form (Review first p.26)

Again, the method is the same for decimals as it is for the whole numbers. If you know the place value well, you can write the number, whole or decimal, in standard form with no work. However, if you need to add the numbers, it is important that you **line up the numbers in column correctly.** For example, write (a) 4,000 + 70 + 3 and (b) 10 + 5 + 0.1 + 0.006 in standard form.

(a)

$$\begin{array}{r} 4,000 \\ 70 \\ +\quad 3 \\ \hline 4,073 \end{array}$$

Standard numeral

For whole numbers, use ones digit as a lead to line up the numbers in column according to their place values.

Add 0s to the empty places.

(b)

$$\begin{array}{r} 10 \\ 5 \\ 0.1 \\ +\ 0.006 \\ \hline \end{array} \qquad \begin{array}{r} 10.000 \\ 5.000 \\ 0.100 \\ +\ 0.006 \\ \hline 15.106 \end{array}$$

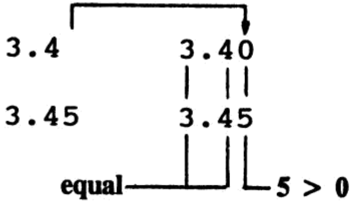
Standard numeral

For decimal numbers, line up the decimal point first. Then add "0s" to the empty places in the columns to make them even in order to avoid errors in computation. (See also p.67)

Comparing and Ordering Decimals (Review first "Symbols for Comparing" p.28)

1. **Compare** 3.4 and 3.45. Use one of the following methods:

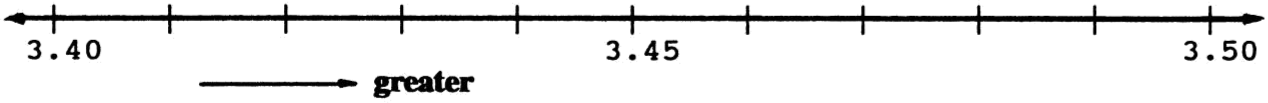
a) **Compare digit-by-digit.**
Write the numbers vertically with **the decimal points lined up** which means to line up the numbers according to their place values. **Add zeros if needed** so all the decimals would have the same numbers of digits. Then compare decimals as you would with whole numbers (See p.29)



Starting from left, compare digits of the same value until you find a pair of digits that are **not equal** - "hundredths" digit.

Since 5 > 0, 3.45 > 3.4 or 3.4 < 3.45

b) **Compare on a number line.**



General rule: On a number line like this, the number *on the right* is *greater* than the number on the left. **So, 3.45 > 3.4** or **3.4 < 3.45**

c) **Compare by using expanded form.**
Write each number as the sum of its digits and compare.

$3.4 = 3.00 + 0.4 + \mathbf{0.00}$

$3.45 = 3.00 + 0.4 + \mathbf{0.05}$

The first place they are not equal is hundredths. **Since 0.05 > 0, 3.45 > 3.4**

> **Note: Be careful in interpreting the result of comparison. For example, if the numbers represent the timing of a race contest, the lesser number won the race. If they represent the distance of frog jumps, the greater number won the contest.**

2. **Ordering from least to greatest:** 3; 0.967; 1.7; 0.96.

Write the numbers vertically with the decimal points lined up. Add zeros if needed so all the numbers have the same number of digits. Compare first the whole number part and then the decimal part, **two numbers at a time.**

3 → 3.000
0.967 → 0.967
1.7 → 1.700
0.96 → 0.960

* Compare ones place: 3 > 1. So, 3 > 1.7

* Compare two decimals. **The first place** that are different is the thousandths. **Since 0.007 > 0, therefore 0.967 > 0.96.**

Order from least to greatest: 0.96 < 0.967 < 1.7 < 3

> **Note: 3 = 3.000; 1.7 = 1.700; 0.96 = 0.960 They are equivalent decimals.**

(See also p.67)

Adding Zeros Before Whole Numbers And Decimals

Whole Numbers

3
03
003
0003
00003

3 remains in "ones" place.

- Adding zeros (0s) **before** a whole number ***does not*** change the value of the number:
  0003 = 003 = 03 = 3

- Therefore, we **drop 0s** that come before a whole number in division when the divisor is larger than the dividend.

Decimals (See also p.69)

.3	3 tenths
.03	3 hundredths
.003	3 thousandths
.0003	3 ten-thousandths
.00003	3 hundred-thousandths

- Adding zeros **immediately after** the decimal point and **before** a decimal digit ***changes*** the value of the decimal.

- **Each zero** you add, ***reduces*** the value of the number to **1/10:**
  - .03 is only 1/10 of .3
  - .003 is only 1/10 of 03

Suggestion: Study this page and next page side by side carefully.

Adding Zeros After Whole Numbers And Decimals

Whole Numbers (See also p.178)

3 — 3
30 — 3 tens
300 — 3 hundreds
3000 — 3 thousands
30000 — 3 ten-thousands

* Adding zeros **after** a whole number ***changes*** the value of the number.

* **Each zero** you add, ***increases*** the value of the number **10 times:**

- 30 is 10 times larger than 3
- 300 is 10 times larger than 30.
- 300 is 100 times larger than 3

Decimals

.3
.30
.300
.3000
.30000

3 remains in the "tenths" place.

* Adding/dropping "0s" **after** a decimal number ***does not*** change the value of the decimal:

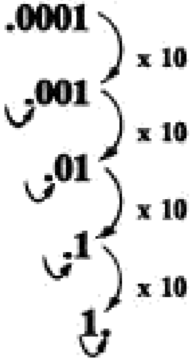
- .3 = .30 = .300 = .3000
- .3000 = .300 = .30 = .3

* Therefore, we add 0s in adding/subtracting/comparing decimals to avoid error.

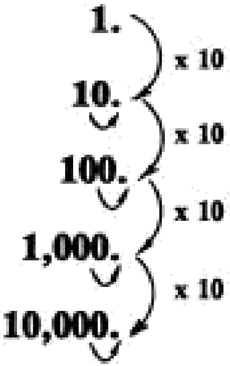
Note: Do you see that the effects on whole numbers and decimals are just opposite?

Multiplying Numbers by 10, 100, etc. & Decimal Places (See also p.228)

Decimals



Whole Numbers



To multiply a number by 10, the decimal point is moved 1 place <u>to the right</u> and the value of the number increases 10 times.

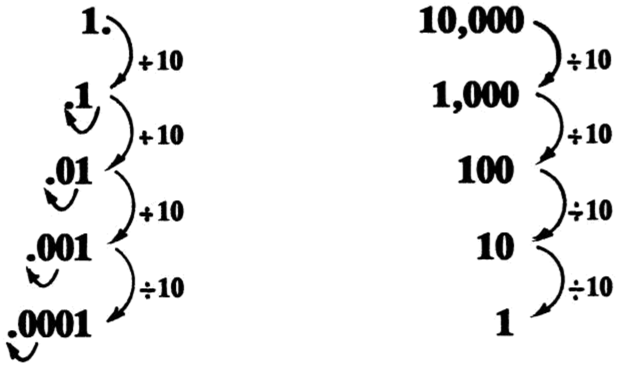
For whole numbers, the decimal point is at the right of ones digit which is usually omitted.

General Rule:

To multiply a decimal:	by **10**	by **100**	by **1000** ...
Move the decimal point **to the right**:	**1** place	**2** places	**3** places ...

Dividing Numbers by 10, 100, etc. & Decimal Places (see also p.229)

Decimals **Whole Numbers**



To divide a number by 10 equals to multiplying by its reciprocal 1/10.

To divide a number by 10, the decimal point is moved 1 place to the left and the value of the number decreases by 1/10.

Use zeros to hold places if necessary.

General Rule:

To divide a decimal:	by **10**	by **100**	by **1000** ...
Move the decimal point **to the left:**	**1** place	**2** places	**3** places ....

Powers of 10 with Negative Exponents (See also "Place Value in Exponents" p.72)

If the numbers found on page 45 **(First Power & Zero Power)** is continued in decreasing order, we will see the following pattern:

The Number		Exponent
10		10^1
	x 1/10	
1		10^0
	x 1/10	
$\frac{1}{10}$		10^{-1}
	x 1/10	
$\frac{1}{100}$		10^{-2}
	x 1/10	
$\frac{1}{1,000}$		10^{-3}
	x 1/10	
$\frac{1}{10,000}$		10^{-4}

Do you observe that when we multiply each successive number by 1/10, which equals to divide by 10, each number is **decreased by a factor of 10.** And it leads to ***negative exponents.***

Powers of 10 with negative exponents can be written as positive exponents *in fraction with 1 as the numerator*:

$$10^{-3} = \frac{1}{10^3}, \quad 10^{-7} = \frac{1}{10^7}$$

Difference Between Positive & Negative Exponents (Read also p.75)

It will help you to see the difference between numbers with positive and negative exponents if we write them in fraction form.

Remember: The exponent indicates the number of times the base is used as a factor.

(a) $$10^3 = \frac{\mathbf{10 \times 10 \times 10}}{1} = 1{,}000$$

With **positive** exponent (3), the factors are **in the numerator.**

$$10^{-3} = \frac{1}{\mathbf{10 \times 10 \times 10}} = \frac{1}{1{,}000}$$

With **negative** exponent (-3), the factors are **in the denominator.**

This is also true of any base other than the base of 10 (powers of 10).

(b) $$3^4 = \frac{\mathbf{3 \times 3 \times 3 \times 3}}{1} = 81$$

With **positive** exponent (4), the factors are **in the numerator.**

$$3^{-4} = \frac{1}{\mathbf{3 \times 3 \times 3 \times 3}} = \frac{1}{81}$$

With **negative** exponent (-4), the factors are **in the denominator.**

Note: 10^{-3} (1/1000) and 10^3 (1000) are **reciprocal** of each other. (See p.326)

Chart: Place Value In Exponents

The value of each place can be expressed in different forms:

Whole numbers Positive exponents				Decimal numbers Negative exponents			
1	1	1	•	1	1	1	**Place value in:**
100	10	1		0.1	0.01	0.001	**Standard form**
$\frac{100}{1}$	$\frac{10}{1}$	$\frac{1}{1}$		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$	**Fraction form**
$\frac{10\cdot10}{1}$	$\frac{10}{1}$	$\frac{1}{1}$		$\frac{1}{10}$	$\frac{1}{10\cdot10}$	$\frac{1}{10\cdot10\cdot10}$	
$\frac{10^2}{1}$	$\frac{10^1}{1}$	$\frac{10^0}{1}$		$\frac{1}{10^1}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	**Exponent form** **Positive** **&** **Negative**
10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}	

The chart below shows the place value of the first five decimal places, the places *to the right* of the decimal point.

In Word	In Decimal Number	In Decimal Fraction	In Exponent
Ones	1	$\frac{1}{1}$	10^{0}
Tenths	0.1	$\frac{1}{10}$	10^{-1}
Hundredths	0.01	$\frac{1}{100}$	10^{-2}
Thousandths	0.001	$\frac{1}{1,000}$	10^{-3}
Ten-thousandths	0.0001	$\frac{1}{10,000}$	10^{-4}
Hundred-thousandths	0.00001	$\frac{1}{100,000}$	10^{-5}

The value of the decimal decreases. ↓

The denominator becomes bigger. ↓

Scientific Notation (Review first "Exponents & Powers of 10," p.39)

Scientific Notation is a convenient way of writing large and small numbers **as a product of two factors** in the following way:

(The first factor must be a number greater than or equal to 1 but less than 10.) **X** **(The second factor is a power of 10 written in exponent form.)**

The **second** factor can be a power of 10 with either:

1. a **positive** exponent - for numbers greater than 10, or
2. a **negative** exponent - for numbers less than 1.

(The above statements are true if the exponents are integers.)

The **first** factor can be either:

1. a **whole number** such as 1, 2, 3, ..., 9, ***but not 10.***
   (Note: 10 can be written in scientific notation by itself as 1 x 10^1.) Or
2. a **mixed decimal** (***not a decimal***) greater than 1 but less than 10.

Example: Are 10.35 x 10^3 and 0.95 x 10^2 written in scientific notation?
Ask yourself: "Are the first factors greater than 1 and less than 10?"
Answer: "No". Neither one is written in scientific notation.

Writing Scientific Notation with Positive & Negative Exponents (Review p.71)

with Positive Exponents

The number given must be **greater than 10.**

Step 1.
Divide the number by a power of 10, reducing it to a **smaller** number, a number between 1 and 10. This is the first factor.

Step 2 is the inverse operation of Step 1.

Step 2.
Multiply the reduced number by the power of 10, by which we divided the number in step 1. This is the second factor.

See examples on pp. 76-77

with Negative Exponents

The number given must be **less than 1.**

Step 1.
Multiply the number by a power of 10, making it a **larger** number, a number between 1 and 10. This is the first factor.

Step 2 is the inverse operation of Step 1.

Step 2.
Divide the number by the power 10, by which we multiplied the number in step 1. This is the second factor.
(To divide by a power of 10 = to multiply by the power of 10 with a negative exponent.)
See an example on p.80

Writing in Scientific Notation - Example 1 shows the process by giving a step-by-step analysis. Examples 2 & 3 use the shortcut.

Example 1: Write 275 (**whole number**) in scientific notation.

$$275. \xrightarrow[\div\ 10]{} 27.5 \xrightarrow[\div\ 10]{} 2.75$$

To divide a number by 10, move the decimal point 1 place to the left. Moving 2 places to the left when dividing it by 100 or 10^2.

$$275 \div 100 = 2.75$$

$$2.75 \times 100 = 275$$

$$\underset{\uparrow}{2.75 \times 10^2}$$

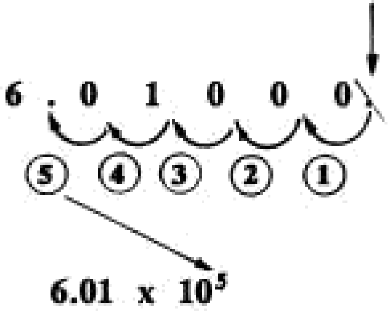
275 in scientific notation

Step 1.
Divide 275 by 10 successively (powers of 10), until 275 is reduced to 2.75, a number between 1 and 10. Write 2.75 as the first factor.

Step 2 is the **inverse operations** of **Step 1.**

Step 2.
Multiply 2.75 by the powers of 10 (10^2) by which we divided the number 275 in step 1. This is the second factor.

Example 2: Write 601,000 (**whole number**) in scientific notation.

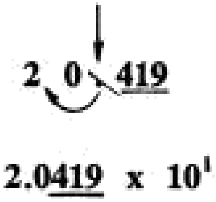


Step 1. Locate the decimal point. It is at the end of "ones" digit.

Step 2. Place the decimal point between 6 and 0 and count how many places the decimal point was moved to the left. The number of places moved is the exponent number.

Step 3. Write 6.01 as the first factor and 10^5 as the second factor. Since 6.01000 = 6.01, drops the 3 zeros.

Example 3: Write 20.419 (**a mixed decimal**) in scientific notation.



- * Locate the decimal point and move it to the place where the number will be reduced to just between 1 and 10.
- * In scientific notation, the 1st factor **must include the decimal part - 419.**
- * 10^1 = 10, the exponent may be omitted.

Changing Scientific Notation To Standard Form (Review first "Powers of 10" p.37)

Example: Write 4.27×10^6 in standard numeral.

4.27×10^6

$$\begin{array}{r} 4.27 \\ \times\ 1{,}000{,}000 \\ \hline 4{,}270{,}000 \end{array}$$

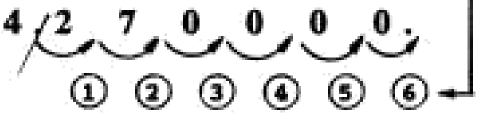
Method 1: Long Way

Multiply 4.27 by 1,000,000 like an arithmetic problem.

← The standard numeral is 4,270,000.

The exponent indicates the number of places the decimal point is to be moved to the right.

4.27×10^6



4,270,000

Method 2: Shortcut

1st, look at the exponent (6). Then moves the decimal point 6 places to the right, add "0" to fill the places.

← The standard numeral is 4,270,000.

Multiplying Numbers by Powers of 10 with Negative Exponents (Review p.71)

Study the following example carefully until you understand what it means to multiply a number by a power of 10 with a negative exponent.

$$5 \times 10^{-2} = 5 \times \frac{1}{10^{2}} = 5 \times \frac{1}{100} = \frac{5}{100} = 5 \div 100 = 0.05$$

To multiply by 10^{-2} equals to divide by 10^{2}

Points To Remember:

* When writing in scientific notation:
  - if the given number is **greater than 10**, the power of 10 has **a positive exponent.**
  - if the given number is **less than 1**, the power of 10 has **a negative exponent.**

* If the exponents are **integers** (whole numbers), multiplying by:
  - a power of 10 with **positive** exponent, makes the number **larger**.
  - a power of 10 with **negative** exponent, makes the number **smaller**.

The above statements are also true of any base(nonzero numbers) other than the base 10, if the exponents are integers.

Writing in Scientific Notation with Negative Exponents

The following example shows you how to write 0.0064, a number less than one, in scientific notation.

Moving the decimal point 3 places to the right = multiplying the number by 1000.

Step 1. Write the first factor.
Multiplying by 1000 (= 10^3) changes 0.0064 to 6.4 -- a larger number, a number greater than 1 and less than 10. Write 6.4 as the first factor.

$$0.0064 \times 10^{3} = \mathbf{6.4}$$

$$\mathbf{6.4} \times 10^{-3} = 0.0064$$

Step 2 is an inverse operation of Step 1. To divide by 1000 equals to multiply by 10^{-3} (1/1000).

$$\mathbf{6.4 \times 10^{-3}}$$

0.0064 in scientific notation

Step 2. Write the second factor.
Multiply 6.4 by 10^{-3} . Multiplying by 10 with a negative exponent (-3) reduces 6.4 to 0.0064, the original number.

Changing Scientific Notation with Negative Exponents to Standard Form (p.37)

The following shows how to write 9.1×10^{-5} in standard numeral.

Remember: To multiply by 10^{-5} is to divide by 10^{5} (= 100,000).

Method 1. Long Way

Divide 9.1 by 100,000 (10^{-5}) as you would divide a decimal.
The standard numeral: **.000091**

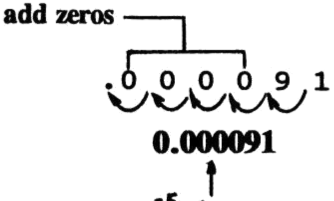
```
             .000091
100000 )9.100000
       - 9 00000
           100000
         - 100000
                0
```

Method 2. Shortcut

First, look at the exponent number. Since negative exponent (-5) means division, move the decimal point 5 places to the left. Fill the places with zeros (0s).

To move the decimal point 5 places to the left is the same as to divide the number by 100,000 (or 10^{5}).

9.1×10^{-5}



9.1×10^{-5} in standard numeral

Summary
(Decimals)

- On a place value chart, the decimal point is the dividing point - to its left are whole numbers and to its right are decimals. The decimal point of a whole number is at the right of the ones digit which is usually omitted.
- Decimals are not grouped into periods of three digits like whole numbers. To read decimals, one must know the names of the decimal places.
- Our money system is similar to the decimal system except it keeps only two decimal places - the tenths (dime) & the hundredths (cent)
- Adding zeros before the whole numbers or after the decimal numbers does not affect the value of the numbers. But adding zeros after the whole numbers, or between the decimal point and a decimal digit changes the value of the number
- To multiply decimals by powers of 10, the decimal points is moved to the right and the number becomes larger; to divide decimals by powers of 10, the decimal point is moved to the left and the number becomes smaller
- Scientific notation is a convenient way of writing very large numbers and very small numbers. If a given number is greater than 10, its power of 10 has a positive exponent; if a given number is less than 1, its power of 10 has a negative exponent.

Part I. Numbers & Concepts

D. Fractions

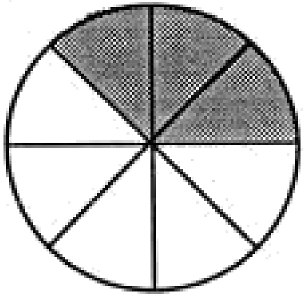
Table of Contents

Meanings Of Fractions (a/b) (Read first p.8)

Keep in mind that numbers written in fraction form (a/b) could be used in one of the following sense:

1. Fractions show **"relationship"** - **"a part of the whole."**

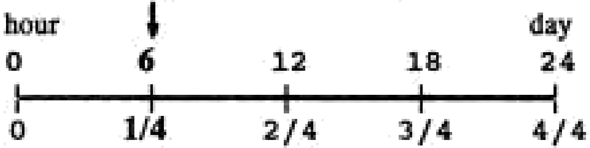
   Example: "A pizza was cut into 8 ***equal*** pieces, and Joe ate 3 pieces."



$$\frac{3}{8}$$

Joe ate 3 pieces
out of
8 EQUAL pieces, the whole

Example: "Sue spent 1/4 of a day in school."



A day was divided into 4 EQUAL length of time. Sue spent one out of 4 EQUAL length of time in school.

In daily life, it is common to use fractions in this sense in dealing with **measurements.** The emphasis is placed on the word **"equal"** - the whole being divided into equal parts.

2. Fractions are used in **"comparing"** two numbers or quantities known as **"ratios."** (See p.372)

Example: "There are 5 girls and 7 boys in the math club. The ratio of girls to boys is:

$$\frac{5}{7}$$

The above fraction, 5/7, can also be expressed as:

5 : 7 or **5 to 7**

Note: Fractions representing ratios are not always treated in the same way as other fractions in computation. (See "Ratios" p.372)

3. Fraction means **"division"** - one of the three ways of writing division (See p.198)

Example: "4 divided by 9" can be written as:

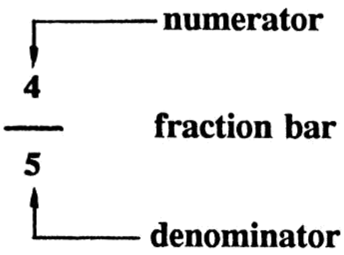
$$\frac{4}{9} \text{ or } 4 \div 9 \text{ or } 9\overline{)4}$$

The quotient of 4 divided by 9 is a decimal. Instead of writing a decimal, we write as a fraction of two whole numbers and call it an **"indicated quotient"** - meaning **division as yet to be carried out.**

Connection: In algebra division is written only in fraction form.

Terms Of A Fraction & Unit Fractions

A fraction, unlike a whole number, is made up of **two numbers** divided by a horizontal line called fraction bar. The top number is called the numerator and the bottom number the denominator. They are the **two terms** of a fraction. That's why we say the fraction is in **lowest terms** when it is in **simplest form**.(p.293)



The numerator tells the number of the equal parts (a part of whole) being involved.

Fraction bar indicates "division" (÷)

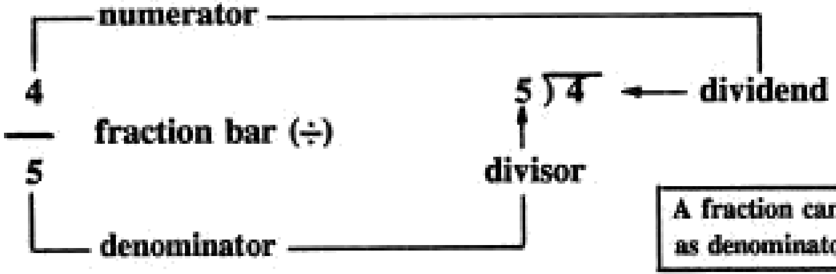
The denominator tells the number of equal parts the whole is divided into.

Unit Fraction: A unit fraction is a fraction with **a numerator of 1:**

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ Unit fractions are proper fractions.

Fractions & Division

Keep in mind that ***fractions means division.*** Fraction form is one of the three ways of writing a division (See p.198) However, in arithmetic computation (not algebra), division is done in ")‾‾" form. The following shows you how to change fractions to division:



A fraction can have any whole number as denominator except zero.

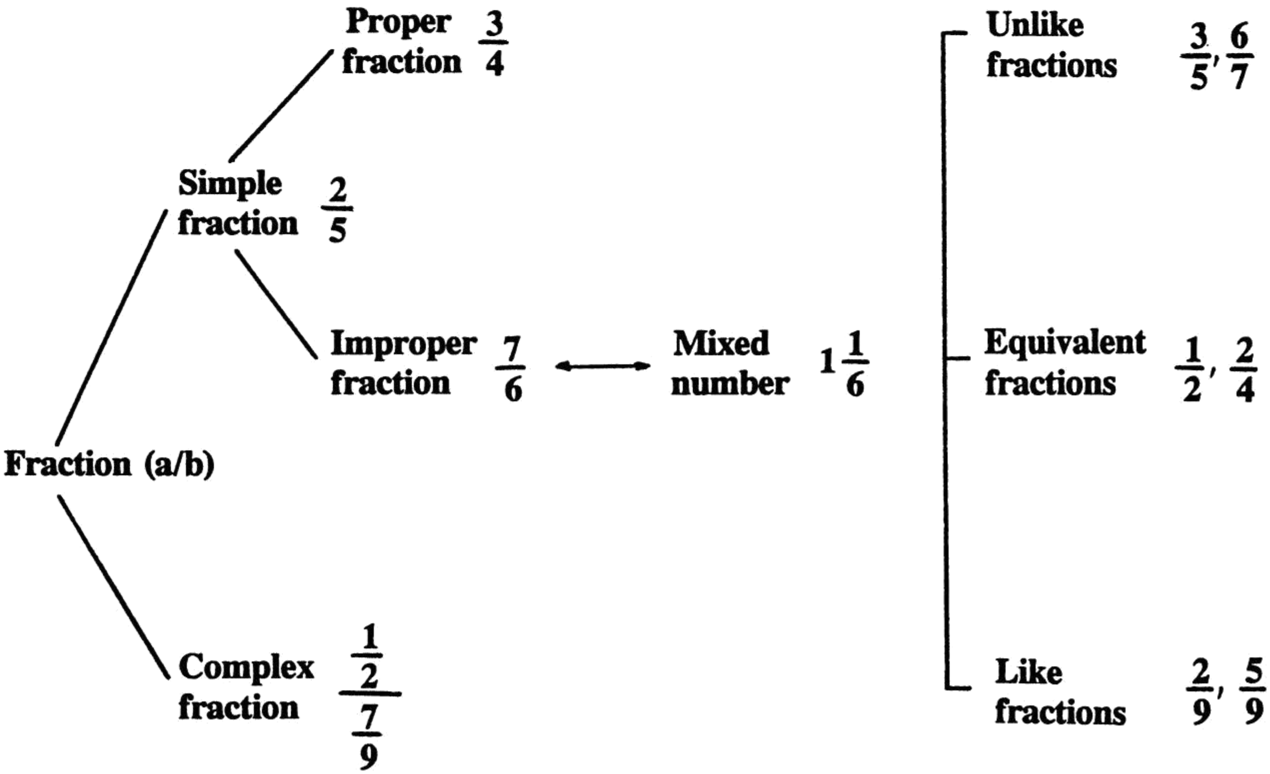
Reading Fractions: Using 4/5 as an example:

1. If it implies **relationship,** read as: **"four fifths."** (concrete number)

When writing fractions in words, the first number is always the numerator

2. If it indicates **division,** read as: **"four divided by five."** (abstract number)

Kinds of Fractions



Definition: Fractions, Simple Fractions, Complex Fractions

$$\frac{a}{b}$$

Fraction - The word "fraction" comes from the Latin *"fractus"* which means *"broken."* That's why some call fractions, the broken numbers. On a number line, the points of fractions (*except* those equals to 1 and other whole numbers) lie between some pair of whole numbers.

$$\frac{2}{5}, \quad \frac{7}{3}$$

Simple Fraction - A simple fraction has ***whole numbers*** for both the numerator and the denominator, excluding "0" as the denominator.

ⓐ ⓑ ⓒ

$$\frac{\frac{1}{2}}{\frac{7}{9}}, \quad \frac{4}{\frac{1}{3}}, \quad \frac{\frac{2}{9}}{6}$$

Complex Fraction - A complex fraction has ***fractions*** for either the numerator or the denominator or both. They are really division of fractions written in fraction form (See p.365). Examples on the left are:

ⓐ A fraction divided by a fraction: 1/2 - 7/9

ⓑ A whole number divided by a fraction: 4 - 1/3

ⓒ A fraction divided by a whole number: 2/9 - 6

Definition: Proper Fractions, Improper Fractions, & Mixed Numbers

$\frac{1}{8}, \quad \frac{1}{2}, \quad \frac{3}{4}$

Proper Fraction - In proper fractions, **the numerator *is smaller* than the denominator.** They are decimals (less than 1) in value.

$\frac{3}{2}, \quad \frac{7}{3}$

Improper Fraction - In improper fractions, the **numerator is either *equal to*** or ***larger than*** **the denominator.** Improper fractions also include:

$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = 1$

a) **One (1)** - When the numerator ***equals*** the denominator **(See p.299)**

$\frac{2}{1} = 2, \quad \frac{3}{1} = 3$

b) **Whole Numbers (2, 3, 4,...)** - When the denominator is 1 **(See p.298)**

$1\frac{1}{2}, \quad 3\frac{2}{5}$

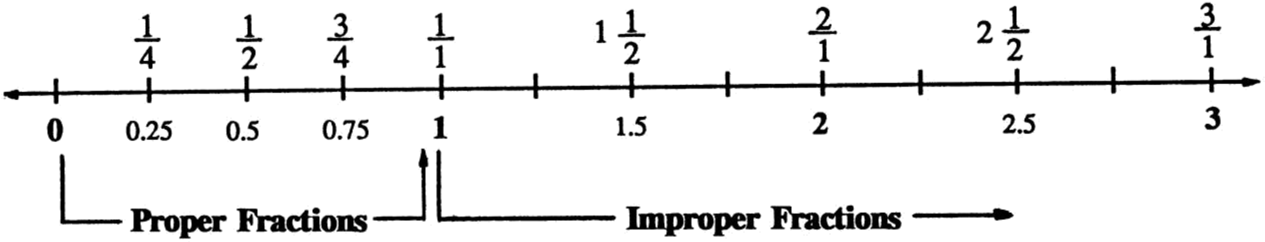
Mixed Number - A mixed number is an **improper fraction** written as **the sum of a whole number and a proper fraction with the plus "+" omitted.**

Example: (an improper fraction) 3/2 = 1 1/2 (a mixed number)

Read the mixed number 1 1/2 as **"one and one half."** **(See p.98)**

Proper Fractions & Improper Fractions on A Number Line (see also p.52)

A number line can help us to see the difference between the proper fractions and improper fractions:



Decimals & Mixed Decimals (see also p.52)

If we **change the fractions to decimals** by dividing the numerator by the denominator, we will notice that:

- **Proper Fractions** are like **decimals** (less than 1). Their points lie between 0 and 1 on the number line as seen above.
- **Improper fractions (Mixed Numbers)** are like **mixed decimals** (greater than 1). Their points lie between some pair of whole numbers on the number line.

Changing Improper Fractions to Mixed Numbers (Read first p.92)

Changing 21/6 to A Mixed Number:

Shortcut

$$\begin{array}{r} 3\ \ 3/6 \\ 6\,\overline{)\,21} \\ -18 \\ \hline 3 \end{array}$$

Divide the numerator by the denominator and write the remainder as a fraction – remainder over divisor – and simplify:
3/6 = 1/2.   21/6 = 3 1/2.

Long Way

$$\frac{21}{6} = \frac{18 + 3}{6}$$

1st. Write the numerator as the sum of two numbers. The first number is the largest multiple of 6 but is smaller than 21. **(6 x 3 = 18, 18 < 21)**

$$\frac{18 + 3}{6} = \frac{18}{6} + \frac{3}{6}$$

2nd. Write it as the sum of two like fractions. **The first fraction is an improper fraction.**
(It's the reverse process of adding like fraction.)

$$\frac{18}{6} + \frac{3}{6} = 3 + \frac{1}{2} = 3\,\frac{1}{2}$$

3rd. Simplify both fractions and write as a mixed number **with (+) sign omitted.**

Changing Mixed Numbers to Improper Fractions

Changing 5 1/2 to An Improper Fraction:

Shortcut

$$5\;\frac{\xrightarrow{+}1}{\xleftarrow[\times]{}2}=\frac{11}{2}$$

Multiply the whole number by the denominator (2) and add the product to the numerator.

Long Way

$$5\ 1/2 = 5 + \frac{1}{2}$$

1st. Write the mixed numbers as whole number + fraction.

$$= \frac{5}{1} + \frac{1}{2}$$

2nd. Write the whole number as a fraction with the denominator of 1.

$$= \frac{5 \times 2}{1 \times 2} + \frac{1}{2}$$

3rd. Change 5/1 to an equivalent fraction with 2 as the denominator.

$$= \frac{10}{2} + \frac{1}{2} = \frac{11}{2}$$

4th. Add the like fractions.

Note: The shortcut omitted step 1 & 2, and combined steps 3 and 4.

Whole Numbers Division & Mixed Numbers (See also pp.89,251)

If you write the *whole number* **divisions** in fraction form, you will notice that every one of them forms an "**improper fraction**" - the numerator larger than the denominator as the following examples show:

(a) $9 \div 3 = \frac{9}{3}$ (b) $14 \div 4 = \frac{14}{4}$ (c) $36 \div 6 = \frac{36}{6}$

When we divide 9 by 3 or 36 by 6, they divide evenly with zero remainder. But when we divide 14 by 4, we get 3 and 2 **left over** which we called ***remainder***. And the quotient of 14 divided by 4 can be written as:

a whole number

$$\begin{array}{r} 3\ \ \text{r2} \\ 4\ \overline{)\ 14} \\ -\ 12 \\ \hline 2 \end{array}$$

a mixed decimals

$$\begin{array}{r} 3.5 \\ 4\ \overline{)\ 14.0} \\ -\ 12 \\ \hline 2\ 0 \\ -2\ 0 \\ \hline 0 \end{array}$$

a mixed number

$$\begin{array}{r} 3\ \ 1/2 \\ 4\ \overline{)\ 14} \\ -\ 12 \\ \hline 2 \end{array}$$

As you see that **mixed numbers** arise often from **division of** ***whole numbers.***

Definition: Unlike Fractions, Like Fractions, Equivalent Fractions

When we deal with unlike fractions or like fractions, we are dealing with **two or more fractions. The difference** between the unlike and like fractions is found in their *denominators.*

$\frac{3}{5} \qquad \frac{6}{7}$

Unlike Fractions - Unlike fractions have **different numbers** for the denominators.

$\frac{2}{9}, \qquad \frac{5}{9},$

Like Fractions - Like fractions have the **same numbers** for the denominators.

$\frac{1}{2}, \quad \frac{2}{4}, \quad \frac{3}{6}, \ldots$

Equivalent Fractions - Equivalent fractions have the **same value** and the **same point** on a number line, but they have *different numbers* for the numerators and for the denominators. Any fraction has a very large set of equivalent fractions. Equivalent fractions are used:

$$\frac{24}{32} = \frac{3}{4}$$

a) in *changing* **unlike** fractions to **like** fractions by using LCM/LCD.

b) in *reducing* a fraction to **lowest terms** by using GCF.

Decimals & Decimal Fractions (Review first "Place Value of Decimals" p.54)

Decimal fractions and decimals are two different ways of writing the same number as you see from the following examples:

a) **Both read "5 tenths"**

$$\underbrace{0.5}_{\text{tenths}} = \frac{5}{10}$$

b) **Both read "4 and 5 hundredths"**

$$\underbrace{4.05}_{\text{hundredths}} = 4\frac{5}{100} \longleftarrow \text{a mixed number}$$

In decimal fractions, a power of 10 (10, 100, etc.) is shown in the denominator. In decimals, a power of 10 is not shown but is indicated by the position of the digit after the decimal point.

c) **Both read "6 thousandths"**

$$\frac{6}{1000} = 0.006$$

Since 6 is in thousandths place (3rd place) use zeros to hold the tenths & hundredths places.

d) **Both read "2 and 6 tenths"**

$$2\frac{6}{10} = 2.6 \longleftarrow \text{a mixed decimal}$$

The decimal point is for the word "and."
2.6 = 2 + 0.6

Changing Decimal Fractions To Decimals: A How-To (See also p.384)

Shortcut: Move the decimal point in the numerator to the left as many places as there are zeros in the denominator. The reason is that **fraction means division.**
Example: Write 642/1000 as a decimals. (See also p.229)

$$\frac{642}{1000} = 642 \div 1000 = .642$$ See "Dividing Decimals by 10, 100, etc."

Changing Decimals To Decimal Fractions: A How-To (See also p.385)

Shortcut: Write the decimal digit as the numerator over 1. Then add to 1 the number of zeros that equal to the number of decimal places. Here is the reason behind the shortcut.
Example: Write .25 as a decimal fraction. (See also p.228)

$$\overset{(1)}{\frac{.25}{1}} = \overset{(2)}{\frac{.25 \times 100}{1 \times 100}} = \overset{(3)}{\frac{25}{100}}$$

The number of zeros in the denominator equals the number of decimal places.

(1) Write the decimal as a fraction - **the decimal over 1.**
(2) Multiply the fraction by 100/100 **to get rid off the decimal point.**
(3) The decimal fraction **25/100** is **an equivalent fraction** of **.25/1.**

Comparing and Ordering Fractions (Read "Equivalent Fractions" p.300)

Fractions can be compared **only** when they have the **same denominators.** For how to change unlike fractions to like fractions see pages 320-325.

1. To compare fractions with **like denominators:** 2/5 and 4/5

$$\frac{2}{5} \qquad \frac{4}{5}$$

If the fractions have the same denominators, **the greater the numerator, the greater the fraction.** Since 4 > 2, 4/5 > 2/5.

2. To compare fractions with **like numerators:** 7/9 and 7/12

$$\frac{7}{9} \qquad \frac{7}{12}$$

If the fractions have the same numerator, **the greater the denominator, the smaller the fraction.** Since 12 > 9, 7/12 < 7/9.

3. To compare fractions with **different denominator:** 3/4 and 5/8

$$\frac{3}{4} \overset{\times 2}{\underset{\times 2}{=}} \frac{6}{8}, \qquad \frac{5}{8}$$

1st. Change unlike fractions to like fractions by using **equivalent fractions.**
2nd. Compare the numerators of the fractions.
Since 6 > 5, 3/4 > 5/8

4. To compare **mixed numbers:** 1 2/3 and 2 7/8

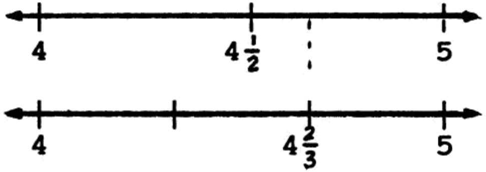
$$1\frac{2}{3},\qquad 2\frac{7}{8}$$

Compare the **whole number part first.**
Since 1 < 2, 1 2/3 < 2 7/8.

5. To compare **mixed numbers:** 4 1/2 and 4 2/3

$$4\frac{1}{2},\qquad 4\frac{3}{6},\qquad 4\frac{2}{3},\qquad 4\frac{4}{6},$$

Since the whole number part are equal, **compare the fractional part.**



Or compare them on **number lines.**
The number on the right is always greater than the number on the left.
So, 4 2/3 > 4 1/2.

6. **To order the numbers** from least to greatest: 3 2/3, 3 2/9, 3 7/9

$$\frac{2}{3} \underset{\times 2}{\overset{\times 2}{=}} \frac{6}{9},\qquad \frac{2}{9},\qquad \frac{7}{9}$$

Compare the whole numbers part first.
Then compare the fractional part.
Order from least to the greatest:
3 2/9 < 3 2/3 < 3 7/9

Summary
(Fractions)

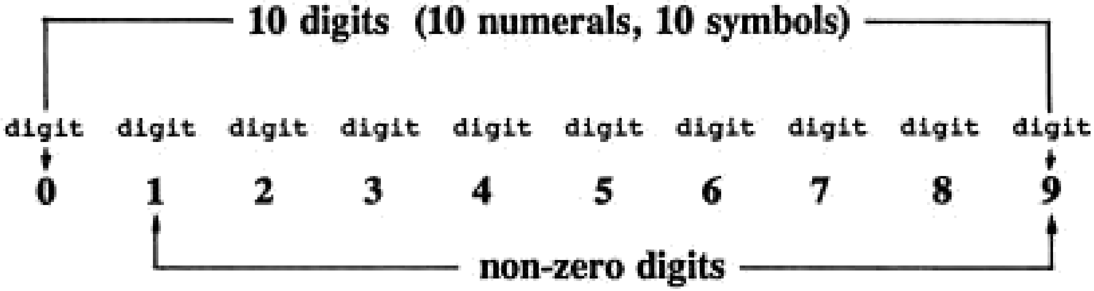
* Fractions are used to express "a part of the whole," or "ratios," or "division." By its context, you can tell which one of the meanings is more relevant to the situation.
* A fraction is made up of two numbers - the numerator and the denominator - which are divided by a horizontal line called a fraction bar
* All fractions signify division. The numerator (top number) is the dividend and the denominator (bottom number) is the divisor
* Fractions can be classified as simple and complex, proper and improper, like and unlike fractions.
* Improper fractions and mixed numbers are interchangeable. Mixed numbers are one of the three ways of writing quotients with remainders in whole number division.
* Decimal fractions can easily be changed to decimals and vice versa.

Part II. Whole Number Operations

A. Introduction

Table of Contents

Digit, Ten Digits



Examples:

10 is a two-digit number.
596 is a **three**-digit number.
27,081 is a **five**-digit number.

Remember, 10 is not a digit.

Connection: We use the ten digits to write numbers, just as we use the twenty-six letters of alphabet to write words. In fact, we can use these ten digits and **a decimal point** to write very large numbers and very small numbers. Remember, we can't write the largest or the smallest numbers. Do you know why? Think!

Meanings Of The Symbol "0"

In the beginning, we were taught that the big "0" (zero) means "nothing." Well, it really depends on where and how the symbol 0 is used. At least, it has the following meanings:

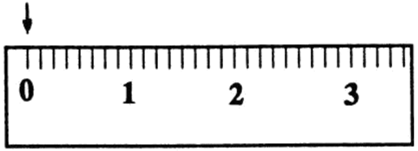
1. $18 - 18 = \overset{\downarrow}{0} \qquad 7 + \overset{\downarrow}{0} = 7$

   Zero means "no thing" (nothing) when it stands by itself.

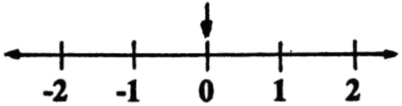
2. $5\overset{\downarrow}{0}9 \qquad 8\overset{\downarrow}{0}\overset{\downarrow}{0}1$

   $\boxed{\text{place holders}}$

   Zero serves as a "place holder" in a number. **When "0s" are used as a place holders, they can not be omitted.**

3. 

   Zero marks the "beginning" in all measurement -- such as rulers, measuring tapes, scales, etc.

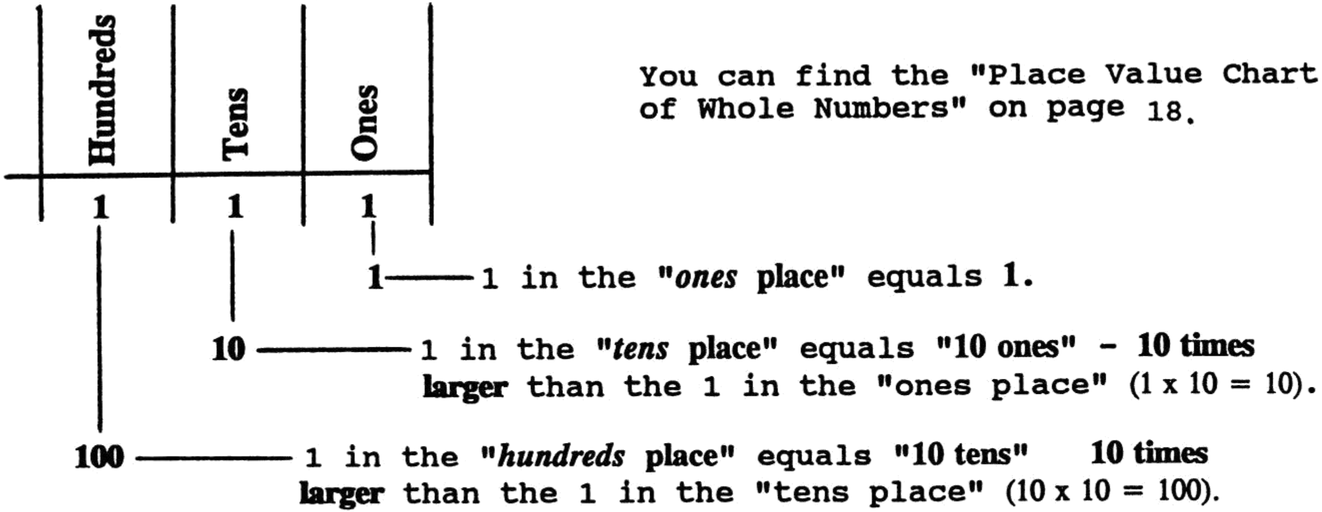
4. 

   Zero is called "origin" on a number line and a coordinate graph.

Decimal System Or Base-10 System (Read also p.16)

Our number system is called decimal system. The word "deci" means "ten." According to the decimal system, **"The value of any place (or digit) is 10 times larger than the place (or digit) to its right."** A place value chart can help us understand how the decimal system works:

You can find the "Place Value Chart of Whole Numbers" on page 18.



1 in the "*ones* place" equals 1.

1 in the "*tens* place" equals "10 ones" - **10 times larger** than the 1 in the "ones place" (1 x 10 = 10).

1 in the "*hundreds* place" equals "10 tens" **10 times larger** than the 1 in the "tens place" (10 x 10 = 100).

Do you know **carrying** and **borrowing** has to do with decimal system? (p.140)

Place Value System (Read also p.17)

The decimal system implies the place value system. Place value system says, **"Every digit in a number has a value, and the value of a digit depends on its place or position in the number."** For example, the number 3 has a different value in each of the following numbers: 3, 39, and 326.

Hundreds	Tens	Ones	
		③	**3** ***ones*** (Read "three")
	③	**9**	**3** ***tens*** 9 ones (Read "thirty-nine")
③	**2**	**6**	**3** ***hundreds*** 2 tens 6 ones (Read "three hundred twenty-six")

Do you see that the number 3 has a different value in each of the numbers above? The value of the number 3 depends on its place in the number. Therefore, **be conscious of place value when you work with numbers!**

Four Basic Operations (See next page for the definition of the terms.)

The four basic operations given below may be considered the foundations of arithmetic. We always **begin with addition**, then subtraction, multiplication, and finally division. The reason for that order is:

- * Subtraction is the ***inverse operation*** of addition. (See p.150)
- * Multiplication is a ***repeated addition***. (See p.168)
- * Division is a ***repeated subtraction***. (See p.192), and the ***inverse operation*** of multiplication. (See p.193)

Addition

6	**addend**
+ 8	**addend**
14	**sum**

Subtraction

14	**minuend**
− 8	**subtrahend**
6	**difference**

Multiplication

5	**multiplicand (factor)**
x 9	**multiplier (factor)**
45	**product (multiple)**

Division

9	**quotient (factor)**
5) 45	**dividend (multiple)**
	divisor (factor)

Mathematics, like every other subject, ***has its own vocabulary.*** If you want to understand the language of the mathematics book, you have to learn ***mathematics terms and expressions.*** The following terms are related to four basic operations. They are used throughout the book. **Memorize them!**

* **Addends** - the numbers that you add together to get sum.
Sum - the answer of adding two or more numbers together.

* **Minuend** - the number from which you subtract (top number).
Subtrahend - the number you use to subtract (bottom number).
Difference (or **remainder**) - the result of subtraction. (The answer)

In subtraction with whole numbers, the minuend is always larger than the subtrahend. (See p.122)

* **Multiplicand** - the number to be multiplied.
Multiplier - the number you use to multiply the multiplicand.
Product (or **multiple**) - the result of multiplication. (The answer)
Factors - the numbers you multiply together to get product.

Factors are another name for the multiplicand, multiplier, divisor, and quotient. (See p.198)

* **Dividend** - the number to be divided into equal part.
Divisor - the number used to divide the dividend.
Quotient - the result of division. (The answer)

In division with whole numbers, the dividend is always larger than the divisor. (See p.122)

Operation Symbols & Key Words

To translate word problems into number sentences (or mathematic equations) is an important part of mathematics study. If you are familiar with the following key words/phrases, it will help you to choose the **correct operation** for solving problems.

Addition: (+)
- 8 **plus** 2 (8 + 2)
- **Add** 6 and 3 (6 + 3)
- 4 **more than** 3 (3 + 4)
- **The sum of** 5 and 7 (5 + 7)
- 2 **increased by** 6 (2 + 6)
- **how many (much) altogether**
- **the total of, in all**

Subtraction: (−)
- 7 **minus** 3 (7 − 3)
- **Subtract** 4 from 9 (9 − 4)
- 10 **less** 5 (10 − 5)
- **The difference of** 8 and 2 (8 − 2)
- 6 **decreased by** 4 (6 − 4)
- **Take away** 3 from 5 (5 − 3)
- **How many (much) are left**

Multiplication: (x)
- 4 **times 6** (4 x 6)
- 5 **multiplied by** 3 (5 x 3)
- **The product of** 2 and 9 (2 x 9)

Division: (÷, $\overline{)\quad}$, /)
- 12 **divided by** 3 (12 ÷ 3)
- **The quotient of** 8 divided by 2
- **Each**

Remember the phrases: **"the sum"**, **"the difference"**, **"the product"**, and **"the quotient"**.

Multiplication Symbols & Division Symbols

It is important for you to know that we can use one of the following symbols to express multiplication or division:

Multiplication Symbols

a. **"x"** (cross-sign)
   Examples. 3 x 6, 2.5 x 0.7, etc.

b. **( )** (parentheses)
   Examples: 4(2.5), (1/3)(5/6), etc.

c. **"·"** (raised dots)
   Examples. 3·6, a·a·a, etc.

d. **"of"** (the word "of")
   Example: a quarter of a day,

Division Symbols

a. "÷"
   Example: 9 ÷ 3, 1 8 ÷ 3

b. "$\overline{)}$"
   Example: $3\overline{)9}$, $3\overline{)18}$

c. " / " (fraction bar)
   Example: 9/3 = 9 ÷ 3

All fractions mean division.

Note: To multiply, we write the problem in **vertical form**

Note: "$\overline{)}$" is the form we use in dividing **whole numbers** and **decimals.**
Since division is not commutative, it is important that you set up the division problem correctly:

- In 9 ÷ 3, the number follow "÷" is the divisor: $3\overline{)9}$
- In 2/5, the denominator is the divisor: $5\overline{)2}$

Checking Addition OR Subtraction

To Check Addition: **Reverse the order of addends and add.**

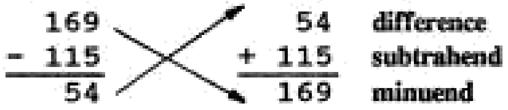
Add	Check
412	375
+ 375	+ 412
787	787

If added down, you check by adding up because addition is **commutative:**

412 + 375 = 787 (added down)
375 + 412 = 787 (added up)

To Check Subtraction: The formula is **(Difference) + (Subtrahend) = (Minuend)**

Subtract **Check**



169 - 115 = 54

To check subtraction, add the difference to the subtrahend. The sum **should equal** the minuend.

We use addition to check subtraction because they are **inverse operations.**

Checking Multiplication OR Division

To Check Multiplication: Reverse the order of the factors and multiply.

Multiply	Check
24	15
x 15	x 24
360	360

Since multiplication is **commutative,** you can multiply in **different order:**

24 x 15 = 360
15 x 24 = 360

To Check Division: The formula is **(Quotient) x (Divisor) = (Dividend)**

Divide	Check	
24	24	**quotient**
15) 360	x 15	**divisor**
- 360	360	**dividend**

360 ÷ 15 = 24

To check the division, multiply the quotient by the divisor, the product **should equal** the dividend.

We use multiplication to check division because they are **inverse operations.** (See also p.198)

Estimating Sums of Whole Numbers (Review first "Rounding Off" p.30)

There are different ways of estimating a sum. The following shows two commonly used methods. Estimate 875 + 1,219 + 2,042:

> Write the numbers vertically according to their place values will help you to determine the front or lead digit of a number Not every first digit of a number is a lead digit like 875 in the following example.

Method 1. Using Front-end Digits

Step 1. Get a **rough estimate** by **adding only the front digits.** "0s" for the other digits.
Step 2. Adjust the estimate (Step 1) by adding an estimated sum of the remaining digits: 875, 219, 42 is about 1,000.

$$\begin{array}{r} 875 \\ 1{,}219 \\ +\ 2{,}042 \\ \hline 3{,}000 \end{array} \qquad \text{about } 1{,}000$$

$$3{,}000 + 1{,}000 = \mathbf{4{,}000}$$

Method 2. Using Rounding

Round each number to the **same place** that can be added easily. So, round to the largest place and add.

$$\begin{array}{rcr} 875 & \longrightarrow & 1{,}000 \\ 1{,}219 & \longrightarrow & 1{,}000 \\ +\ 2{,}042 & \longrightarrow & 2{,}000 \\ \cline{1-1} 4{,}136 & & 4{,}000 \\ \text{actual sum} & & \text{estimate sum} \end{array}$$

> Remember: Use front-end digits estimation (Step 1) without adjustment (step 2) will lose too much especially when the numbers are large. Adjusting the estimate is an attempt to get closer to the actual sum or difference.

Estimating Differences of Whole Numbers (Review first "Rounding Off" p.30)

The following shows how to estimate the difference of 24,249 - 4,888:

The front digit (4) of the smaller number (4,888) is the lead digit.

$$\begin{array}{r} 21{,}249 \\ -\ 4{,}888 \\ \hline \end{array} \qquad \begin{array}{r} 21{,}000 \\ -\ 4{,}000 \\ \hline 17{,}000 \end{array}$$

Method 1. Using Front-end Estimation

Step 1. Get a **rough estimate** by subtracting the front digits: 17,000

Since 888 > 249, the difference is < 17,000

Step 2. Compare the remaining digits and **adjust** the difference.

$$\begin{array}{r} 21{,}249 \\ -\ 4{,}888 \\ \hline \end{array} \quad \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \end{array} \quad \begin{array}{r} 20{,}000 \\ -\ 0{,}000 \\ \hline 20{,}000 \end{array}$$

Method 2. Using Rounding

Round each number to the **same place**, the largest place: ten-thousands.

The above estimate is **not reasonable** because 4,888, the subtrahend, was rounded down to 0. Try again.

$$\begin{array}{r} 21{,}249 \\ -\ 4{,}888 \\ \hline \end{array} \quad \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \end{array} \quad \begin{array}{r} 21{,}000 \\ -\ 5{,}000 \\ \hline 16{,}000 \end{array}$$

So, round to the next largest place: to the nearest thousands.

← This estimate is **more reasonable.**

Estimating Products of Whole Numbers (Review first "Rounding Off" p.30)

There are many ways to estimate a product. Choose a method that will give a closer estimate for the situation. Each method involves multiplying the multiples of 10, 100, etc. (See also p.180)

$$\begin{array}{r} 34 \\ \underline{\times\ 27} \end{array} \qquad\qquad \begin{array}{r} 30 \\ \underline{\times\ 20} \\ 600 \end{array}$$

Method 1. Using Front Digits

Multiply only the front digits. The estimate is always **an underestimate.**

$$\begin{array}{rcccr} 34 & \longrightarrow & \text{down} & \longrightarrow & 30 \\ \underline{\times\ 27} & \longrightarrow & \text{up} & \longrightarrow & \underline{\ \ 30} \\ &&&& 900 \end{array}$$

Method 2. Rounding One down & One Up

Round one factor down, the other up and multiply. It gives a **closer estimate.**

$$\begin{array}{rcr} \multicolumn{3}{c}{\textbf{Round down}} \\ 34 & \longrightarrow & 30 \\ \underline{\times\ 27} & \longrightarrow & \underline{\ \ 20} \\ && 600 \end{array} \qquad\qquad \begin{array}{rcr} \multicolumn{3}{c}{\textbf{Round up}} \\ 34 & \longrightarrow & 40 \\ \underline{\times\ 27} & \longrightarrow & \underline{\ \ 30} \\ && 1200 \end{array}$$

Method 3. Find the Range of a Product

Rounding **both factors** ***up***, and **both** ***down*** gives **the range of the product.** The actual product will be **between** 600 **and** 1200.

$$\begin{array}{rcr} 572 & \longrightarrow & 600 \\ \underline{\times\ \ 63} & \longrightarrow & \underline{\ \ 60} \\ && 3600 \end{array}$$

Method 4. Rounding to the Largest Place

Round each number to **its largest place, not the same place,** and multiply.

Estimating Quotients of Whole Numbers

One way to estimate quotients is to use the multiplication/division facts and powers of 10. We change the given dividend, or divisor, or both to a pair(s) of multiplication fact(s) which are **compatible to the given numbers** so that we can **divide mentally.**

a) $$227 \div 4 \begin{cases} 200 \div 4 = 50 \\ 240 \div 4 = 60 \end{cases}$$

Since $5 \times 4 = 20$ & $6 \times 4 = 24$, **change** 227 to 200 ($20 \div 4$) or 240 ($24 \div 4$) - both numbers are close to 227. The actual quotient is between 50 and 60.

b) $$472 \div 55 \begin{cases} 450 \div 50 = 9 \\ 500 \div 50 = 10 \end{cases}$$

Since $9 \times 5 = 45$ & $10 \times 5 = 50$, **change** 55 to 50 and change 472 to 450 or 500 -- both numbers are close to 472.

$472 \div 55$ $480 \div 60 = 8$

Or **change** 55 to 60 and change 472 to 480 because $48 \div 6 = 8$.

Note: The above examples show that there can be **more than one** reasonable estimates because there are more than one pair of compatible numbers. However, you should have some idea whether **the estimate is an underestimate or overestimate and know the reason why.**

Four Basic Operations & Properties

The properties of mathematics are something that are always true and never changed. If you want to do well in math, it is **a must** that you study carefully the properties and learn how to use them to simplify computations. Some of the basic properties are listed below for your convenience. To know how a property is used, please turn to the indicated page. In order to write these properties in general form, we use a, b, and c, to represent **any whole numbers.**

Addition Properties:

1. **Identity (Zero) Property:** $a + 0 = a$
   The sum of a number and zero is that number. **(See p.130)**
2. **Commutative (Order) Property:** $a + b = b + a$
   Changing the order of the addends does not change the sum. **(See p.130)**
3. **Associative (Grouping) Property:** $(a + b) + c = a + (b + c)$
   Re-grouping of addends does not affect the sum. **(See p.134)**

Subtraction Property of Zero:

1. The difference of a number and zero is the number: $a - 0 = a$ **(See p.153)**
2 The difference of a number and itself is zero: $a - a = 0$ **(See p.153)**

Connection: These properties are also applied to ***Decimals*** & ***Fractions***.

Multiplication Properties:

1. **Zero Property:** $a \times 0 = 0$
   The product of any number and zero is zero. (See p.172)
2. **Identity (One) Property:** $a \times 1 = a$
   The product of any number and one is that number. (See p.172)
3. **Commutative (Order) Property:** $a \times b = b \times a$
   Changing the order of factors does not affect the product. (p.172)
4. **Associative (Grouping) Property:** $(a \times b) \times c = a \times (b \times c)$
   Re-grouping of factors does not affect the product. (p.202)
5. **Distributive Property of Multiplication:** (See p.137)
   $a \times (b + c) = (a \times b) + (a \times c)$ and $(b + c) \times a = (b \times a) + (c \times a)$
   $a \times (b - c) = (a \times b) - (a \times c)$ and $(b - c) \times a = (b \times a) - (c \times a)$

Division Properties:

1. **Zero Property:** $0 \div a = 0$ (a can not be zero) (See p.195)
   The quotient of zero divided by any number is zero. (See p.196)
2. **One Property:** $a \div a = 1$ (a can not be zero)
   The quotient of any number divided by itself is one. (See p.196)
3. **Identity Property:** $a \div 1 = a$
   The quotient of any number divided by one is that number. (p.196)
4. **Division by zero:** $a \div 0$ & $0 \div 0$ **is impossible!** (See p.195)

Connection: One property of multiplication/division is important in fractions.

Special Rules For Subtraction And Division

Keep the following rules in mind when you do **subtraction** and **division.**

1. In **subtracting** *whole numbers, decimals, and fractions,* we always **subtract a smaller number from a larger number.** For example:

   $7 - 4 = 3$ and $4 - 7 =$ **impossible** (because $4 - 7 = -3$)

   $3.5 - 2.1 = 1.4$ and $2.1 - 3.5 =$ **impossible** ($2.1 - 3.5 = -1.4$)

   $5/7 - 2/7 = 3/7$ and $2/7 - 5/7 =$ **impossible** ($2/7 - 5/7 = -3/7$)

   If we subtract a larger number from a smaller number, the answer will be a **"negative"** number. **In arithmetic, we are dealing only with "positive" numbers.**

2. In **dividing** *whole numbers,* **the divisor must be smaller, or at most equals to, the dividend.** For example:

   $5 \div 4 = 1$ r1 and $4 \div 5 =$ **impossible** (because $4 \div 5 = 0.8$)

   If we divide a smaller number by a larger number like $4 \div 5$, **the answer will not be a "whole number" but a "decimal", less than 1.**

Binary Operation & Basic Facts

A binary operation is an operation that performs on two numbers at a time. **The four basic operations are binary operations, which means you add, subtract, multiply, or divide only two numbers at a time.** See the examples below:

1. Add. 8 + 6 + 7 + 9 + 5
   First. You add two numbers, any two, together: 8 + 6 = 14
   Then add another number to the sum of 14: 14 + 7 = 21
   And continue to add another number to the new sum,...

2. Subtract.

$$\begin{array}{r} 69 \\ -\ 25 \\ \hline 44 \end{array}$$

1st. You subtract the ones digits: 9 - 5 = 4.
2nd. You subtract the tens digits: 6 - 2 = 4.
(6 - 2 = 4 is actually 60 - 20 = 40)

2. Multiply.

$$\begin{array}{r} 521 \\ \times\ \ 4 \\ \hline 2084 \end{array}$$

1st. You multiply 1 x 4 = 4.
2nd. You multiply 2 x 4 = 80 (actually 4 x 20)
3rd. You multiply 5 x 4 = 2000 (actually 4 x 500)

Look over the examples again, do you see that we worked on two numbers at a time, and each time we used the basic facts? The fact is, **you will use the basic facts to compute all** addition, subtraction, multiplication, and division problems, no matter how large the numbers may be.

Summary

(Introduction)

* We use the ten digits, with zeros (0s) as place holders, to write numbers. When zeros are used as place holders, they can not be omitted.
* With a sound understanding of the decimal system/place value system, many arithmetic works become easy.
* The four basic operations - addition, subtraction, multiplication, and division - are different ways of counting and are related. For example, We can use addition to check subtraction, and use multiplication to check division.
* Knowledge of mathematical vocabulary and symbols are essential in studying math.
* Properties are the rules of mathematics. They are used to simplify computations. Learn those side by side with the four basic operations and use them to your advantage.
* Rounding is a common skill used to estimate the answers of computations.

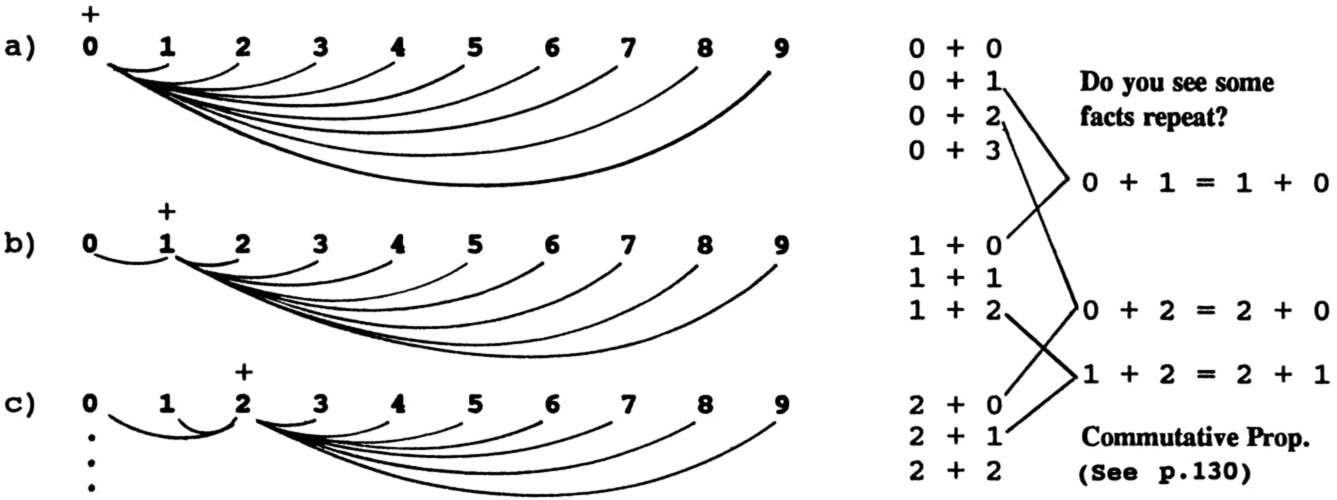
Part II. Whole Number Operations

B. Addition

Table of Contents

Addition Facts: Combinations of Ten Digits (See "Ten Digits" p.106)

The following shows how we got our addition facts. They are the combinations of the ten digits (0 to 9). Start with 0. Add 0 to itself, then added to each of the remaining digits, one at a time in order, as seen in a). Follow the same procedure for each digit, 1 through 9, and you will have the 100 basic addition facts found on the next page.



Do you see some facts repeat?

Commutative Prop. (See p.130)

Note: The sign "+" serves to remind you that you always add the number to itself.

100 Basic Addition Facts - One-Digit Addition

0 + 0 = 0
0 + 1 = 1
0 + 2 = 2
0 + 3 = 3
0 + 4 = 4
0 + 5 = 5
0 + 6 = 6
0 + 7 = 7
0 + 8 = 8
0 + 9 = 9

1 + 0 = 1
1 + 1 = 2
1 + 2 = 3
1 + 3 = 4
1 + 4 = 5
1 + 5 = 6
1 + 6 = 7
1 + 7 = 8
1 + 8 = 9
1 + 9 = 10

2 + 0 = 2
2 + 1 = 3
2 + 2 = 4
2 + 3 = 5
2 + 4 = 6
2 + 5 = 7
2 + 6 = 8
2 + 7 = 9
2 + 8 = 10
2 + 9 = 11

3 + 0 = 3
3 + 1 = 4
3 + 2 = 5
3 + 3 = 6
3 + 4 = 7
3 + 5 = 8
3 + 6 = 9
3 + 7 = 10
3 + 8 = 11
3 + 9 = 12

4 + 0 = 4
4 + 1 = 5
4 + 2 = 6
4 + 3 = 7
4 + 4 = 8
4 + 5 = 9
4 + 6 = 10
4 + 7 = 11
4 + 8 = 12
4 + 9 = 13

5 + 0 = 5
5 + 1 = 6
5 + 2 = 7
5 + 3 = 8
5 + 4 = 9
5 + 5 = 10
5 + 6 = 11
5 + 7 = 12
5 + 8 = 13
5 + 9 = 14

6 + 0 = 6
6 + 1 = 7
6 + 2 = 8
6 + 3 = 9
6 + 4 = 10
6 + 5 = 11
6 + 6 = 12
6 + 7 = 13
6 + 8 = 14
6 + 9 = 15

7 + 0 = 7
7 + 1 = 8
7 + 2 = 9
7 + 3 = 10
7 + 4 = 11
7 + 5 = 12
7 + 6 = 13
7 + 7 = 14
7 + 8 = 15
7 + 9 = 16

8 + 0 = 8
8 + 1 = 9
8 + 2 = 10
8 + 3 = 11
8 + 4 = 12
8 + 5 = 13
8 + 6 = 14
8 + 7 = 15
8 + 8 = 16
8 + 9 = 17

9 + 0 = 9
9 + 1 = 10
9 + 2 = 11
9 + 3 = 12
9 + 4 = 13
9 + 5 = 14
9 + 6 = 15
9 + 7 = 16
9 + 8 = 17
9 + 9 = 18

Addition Properties & Addition Facts (Review first "Properties" p.120)

It is **very** important that you memorize the addition facts because all future addition problems and other operations, depend on these basic facts. The following shows that by using **two addition properties,** you can cut down the memory work by half.

$0 + 1 = 1$ or $1 + 0 = 1$
$0 + 2 = 2$ $2 + 0 = 2$
$0 + 3 = 3$ $3 + 0 = 3$
$0 + 4 = 4$ $4 + 0 = 4$
$0 + 5 = 5$ $5 + 0 = 5$

The **zero property** of addition says, **"Any number plus 0 is the number."** If you know this property, you can omit 19 facts from the list - they are the ones inside the boxes. (See p.129)

$0 + 1 = 1 + 0$
$3 + 5 = 5 + 3$
$4 + 7 = 7 + 4$
$6 + 8 = 8 + 6$
$9 + 2 = 2 + 9$

The **Commutative property** of addition says, **"The order in which numbers are added does not affect the result."** Again, by using this property, we cut down the numbers of facts to be memorized by half. (See next page.)

Remember: These two addition properties apply also to other addition problems including decimals and fractions.

Rearranging The Addition Facts

By using the **zero property** and the **commutative property** of addition, the numbers of addition facts to be learned are reduced to 45 as listed below. In practicing, **make it a habit** of saying each addition facts **both ways** before giving the sum like this: **3 + 5 = 5 + 3 = 8** or **3 + 5 equals 5 + 3 is 8.**

* 1 + 1 = 2
1 + 2 = 2 + 1 = 3
1 + 3 = 3 + 1 = 4
1 + 4 = 4 + 1 = 5
1 + 5 = 5 + 1 = 6
1 + 6 = 6 + 1 = 7
1 + 7 = 7 + 1 = 8
1 + 8 = 8 + 1 = 9
1 + 9 = 9 + 1 = 10
* 2 + 2 = 4
2 + 3 = 3 + 2 = 5
2 + 4 = 4 + 2 = 6
2 + 5 = 5 + 2 = 7
2 + 6 = 6 + 2 = 8
2 + 7 = 7 + 2 = 9
2 + 8 = 8 + 2 = 10
2 + 9 = 9 + 2 = 11
* 3 + 3 = 6
3 + 4 = 4 + 3 = 7
3 + 5 = 5 + 3 = 8
3 + 6 = 6 + 3 = 9
3 + 7 = 7 + 3 = 10
3 + 8 = 8 + 3 = 11
3 + 9 = 9 + 3 = 12
* 4 + 4 = 8
4 + 5 = 5 + 4 = 9
4 + 6 = 6 + 4 = 10
4 + 7 = 7 + 4 = 11
4 + 8 = 8 + 4 = 12
4 + 9 = 9 + 4 = 13
* 5 + 5 = 10
5 + 6 = 6 + 5 = 11
5 + 7 = 7 + 5 = 12
5 + 8 = 8 + 5 = 13
5 + 9 = 9 + 5 = 14
* 6 + 6 = 12
6 + 7 = 7 + 6 = 13
6 + 8 = 8 + 6 = 14
6 + 9 = 9 + 6 = 15
* 7 + 7 = 14
7 + 8 = 8 + 7 = 15
7 + 9 = 9 + 7 = 16
* 8 + 8 = 16
8 + 9 = 9 + 8 = 17
* 9 + 9 = 18

Note: The addition facts with * have identical addends.

Addition Table

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

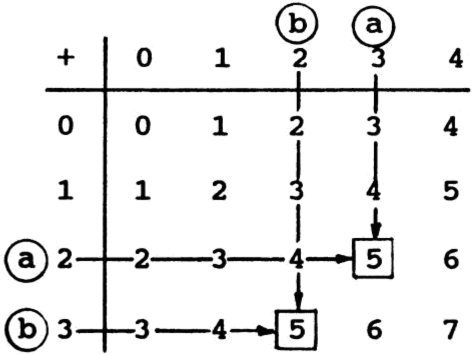
← Addends

↑ Addends

Study the table carefully. Do you see the addition properties? Do you see different pairs of numbers that add up to the same number?

Using The Addition Table To Find Sums

In the addition table, the digits at the top and the digits at the left side are addends. There are **two ways** of looking up the sum of two numbers because ***additions are commutative***. See the example below:



+	0	1	ⓑ 2	ⓐ 3	4
0	0	1	2	3	4
1	1	2	3	4	5
ⓐ 2	2	3	4	5	6
ⓑ 3	3	4	5	6	7

Finding The Sum: 3 + 2 = 2 + 3 = □

- * Find 3 in the top row ⓐ, and 2 in the left column ⓐ. Then go down the column of 3 and go across the row of 2. **The sum is the number where the column and the** row meet. Or
- * Find 3 in the left column ⓑ, and 2 in the top row ⓑ. **The sum is where the column and the row meet.**

Patterns In Addition Table

Study carefully the addition table and see whether you can find the following:

- * Do you see all the addition properties?
- * Do you see the different pairs of numbers that give the same sum?

Numbers Adding Up To 10 (See also "Subtracting From 10" p.155)

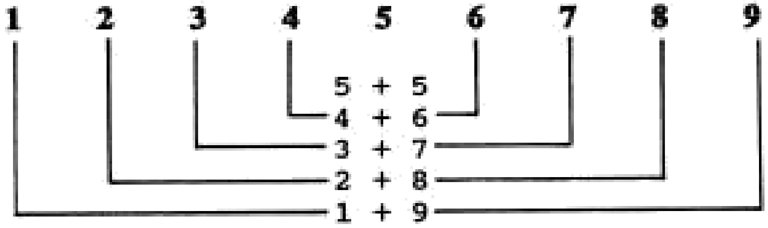
When adding a column of numbers, **always look for numbers, two or three, that add up to 10.** It helps to speed up the computations. See the examples below:

$$\text{a)}\ \begin{array}{r} 4\\ 3\\ 6\\ +\ 7\\ \hline 20 \end{array} \qquad \text{b)}\ \begin{array}{r} 7\\ 5\\ 2\\ +\ 1\\ \hline 15 \end{array} \qquad \text{c)}\ \begin{array}{r} 16\\ 6\\ 3\\ +\ 4\\ \hline 29 \end{array} \qquad \text{d)}\ \begin{array}{r} 1\\ 8\\ 9\\ +\ 2\\ \hline 20 \end{array}$$

In a) 4 + 6 = 10; 3 + 7 = 10
b) 7 + 2 + 1 = 10
c) 16 + 4 = 20
d) 1 + 9 = 10; 8 + 2 = 10
Here we use the associative property

Use the following diagram to help you **memorize** the two numbers and three numbers that add up to ten.

Two numbers add up to 10:



Three numbers add up to 10:

1 + 1 + 8 = 10
1 + 2 + 7 = 10
1 + 3 + 6 = 10
1 + 4 + 5 = 10
2 + 2 + 6 = 10
2 + 3 + 5 = 10
2 + 4 + 4 = 10
3 + 3 + 4 = 10

Applying What You Know

Put your brain to work when studying mathematics! When doing math work, take a good look at the problem first. Always look for ways that you can apply what you already know **to simplify the computation.** For example, if you know the numbers that add up to 10, then you can compute the following problems mentally.

a) If you know: 2 + 8 = 10 **Remember, you can break numbers apart!**
Then apply: 12 + 8 = 20 **Think:** 32 = 30 + 2; 18 = 10 + 8.
32 + 18 = 50 ← So, (32 + 8) + 10 = 50, or
30 + (2 + 18) = 50

b) If you know: 2 + 5 = 7
Then apply: 20 + 50 = 70
200 + 500 = 700

Check:

$$\begin{array}{r} 20 \\ +\ 50 \\ \hline 70 \end{array} \qquad \begin{array}{r} 200 \\ +\ 500 \\ \hline 700 \end{array}$$

If the addends have equal numbers of zeros at the end, add the non-zero digits and keep the number of zeros.

c) If you know: 4 + 6 = 10
Then apply: 40 + 60 = 100
400 + 600 = 1000

Check:

$$\begin{array}{r} 40 \\ +\ 60 \\ \hline 100 \end{array} \qquad \begin{array}{r} 400 \\ +\ 600 \\ \hline 1000 \end{array}$$

Playing With Numbers (I) - Breaking Numbers Apart

In the process of learning the basic facts, many students tend to see these facts as something fixed. In fact, **any digit or number larger than 1 can be broken apart into a sum of two smaller numbers.** If you know the **rules of mathematics,** you can manipulate numbers with ease. For example:

a) 2 + 5 = 7. **Remember 7 can also be written as: 0 + 7, 1 + 6, 3 + 4.**

2 + 5 = 3 + ☐ **Replace 7 with 3 + 4.** Then cover 4 and ask your friend to **find the missing addend.** ***Fun!***

2 + 5 = 5 + ☐ How about **replace 7 with 5 + 2.** Cover 2 and ask your friend to **name the property!** ***Fun!***

b) 8 + 5 = ☐ You can break either 8 or 5 apart into two numbers, so there will be **two numbers that add up to 10.** ***Speed up!***

8 + 5 = ☐ **Think:** 8 = **3 + 5.** So, 3 + **(5 + 5)** = 3 + **10** = 13
8 + 5 = ☐ **Think:** 5 = **2 + 3.** So, **(8 + 2)** + 3 = **10 +** 3 = 13

Rule: **You always compute the numbers inside the parentheses first.**
So, put the numbers that add to 10 inside the parentheses.

c) $13 - 4 = \square$ Break 13 apart. **subtract 4 from 10.** Then add the difference to the ones digit. (See p.155)

$10 + 3 - 4 =$ **Think: 13 = 10 + 3.**

$(10 - 4) + 3 =$ **Rearrange the numbers: Subtract 4 from 10.**

$\downarrow$

$6 + 3 = 9$ Add the difference to 3. *Easy!*

d) $5 \times 27 = \square$ If you know that you can break numbers apart, you can solve the problem in your head.

$5 \times (20 + 7) =$ **Think:** 27 = **20 + 7,** because 5 x 20 is easy.

$(5 \times 20) + (5 \times 7) =$ **Distributive property over addition.**

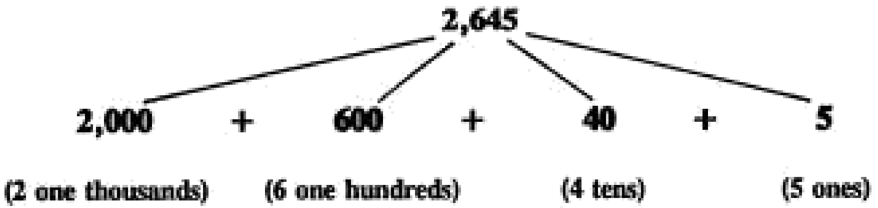
$\downarrow \qquad \downarrow$

$100 + 35 = 135$ Multiply 5 x 27, you will get 135.

The examples given above are only a sample of what you can do with numbers. If you know numbers can be taken apart, you can speed up computations and turn many problems into **mental math.** *Math is fun!*

Writing Numbers in Expanded Form (See also "Expanded Form" p.24)

To write a number in expanded form is to write it **as the sum of the value of each digit.** The following shows 2645 in expanded form:



$$\begin{array}{r} 2{,}000 \\ 600 \\ 40 \\ +\quad 5 \\ \hline 2{,}645 \end{array}$$

A number can be expressed in various forms. We can write 2645
a) **in words:** **"two thousands, six hundred forty-five"**
b) **in standard numeral:** **2,645 (or 2645)**
d) **in expanded form:** **2,645 = 2,000 + 600 + 40 + 5**

It is important that you learn to read and write numbers correctly, especially large numbers. You can find the instructions on pages 18-19.

Addition Using Expanded Form (Review first the previous page.)

Writing addition problems in expanded form can help us to see the process of addition. First, write each addend in an expanded form.

Example: Add. 85 + 67

$$\begin{array}{rcrl} 85 & = & 80 + 5 & \\ +\,62 & = & 60 + 2 & \\ \cline{1-1}\cline{3-3} 7 & & 140 + 7 & = 147 \\ +\,140 & & & \\ \cline{1-1} 147 & & & \end{array}$$

Writing each addend in expanded form helps us to see, that when we add **8 + 6 = 14,** we are actually adding **80 + 60 = 140** with **0 in the ones place omitted.** Do you know Why?

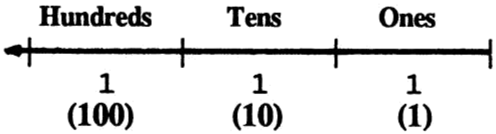
Example: Add. 8 + 97 + 103

$$\begin{array}{rcrl} 1\,1 & & & \\ 8 & & 8 & \\ 97 & = & 90 + 7 & \\ +\,103 & = & 100 + 00 + 3 & \\ \cline{1-1}\cline{3-3} 208 & & 100 + 90 + 18 & = 208 \end{array}$$

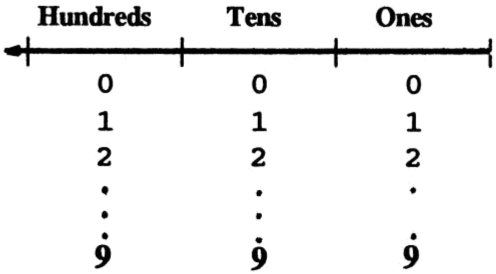
Make sure the place value of the digits are **lined up correctly.**
Then,

- Add the ones: 8 + 7 + 3 = 18. ***Carry*** **one 10 to the tens place by adding 1 to the tens.**
- Add the tens: 10 + 90 = 100. ***Carry*** **one 100 to the hundreds place.**

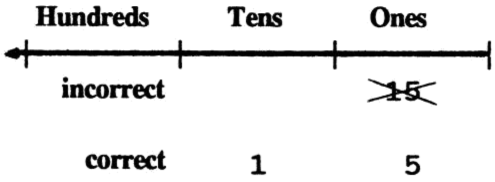
Carrying In Addition & Decimal System



Remember that in decimal system or base 10 system the value of each place is **10 times larger** than the place to its right.



According to base 10 system, each place (ones, tens, hundreds, etc.) can have ***only*** **one-digit numbers: 0, 1, 2, 3, ...** ***up to 9.*** That means we ***can not*** have **two-digit numbers** like 10, 11, 12..., etc. in any place. See the example below. Because of this, we have "**carrying**" in addition.



Therefore, in the process of adding numbers, **if the sum of any place (ones, tens, hundreds, etc.) is 10 or more,** we have to ***carry*** the 10, 20, etc. ***to the next higher place on the left.***

(Continued on the next page.)

Examples Of Carrying In Addition (First, read the previous page.)

(a)

$$\begin{array}{crr} & & 4 \\ + & & 5 \\ \hline & & 9 \end{array}$$

(b)

$$\begin{array}{crr} & & 2 \\ + & & 8 \\ \hline & 1 & 0 \end{array}$$

(c)

$$\begin{array}{crr} & 1 & \\ & & 7 \\ + & 2 & 4 \\ \hline & 3 & 1 \end{array}$$

(d)

$$\begin{array}{crr} & 2 & \\ & & 9 \\ & & 6 \\ + & 4 & 8 \\ \hline & 6 & 3 \end{array}$$

(e)

$$\begin{array}{crrr} & & 5 & 5 \\ + & & 9 & 2 \\ \hline & 1 & 4 & 7 \end{array}$$

(a) Adding *ones* place: 4 + 5 = 9. *No carrying.*

(b) Adding *ones* place: 2 + 8 = 10. *Carrying.*
10 ones = 1 ten. So, write 1 in the tens place & 0 in the ones place.

(c) Adding *ones* place: 7 + 4 = 11. **Carrying.**
11 = 10 + 1 Carry 10 ones to the tens place by adding 1 (10 + 20 = 30)

(d) Adding *ones* place: 9 + 6 + 8 = 23. *Carrying.*
23 = 20 + 3. Carry 20 ones to the tens place by adding 2. (20 + 40 = 60)

(e) Adding *tens* place: 5 + 9 = 14. *Carrying.*
14 = 10 + 4. Carry 10 tens to hundreds place by adding 1

Note: We use the word "carrying" only when the place we carry to has numbers like (c) & (d).

Writing Numbers In Vertical Form (Read first "Place Value" p.18)

Math problems are often written horizontally to save space like the following examples:

(a) 2 + 4 + 6 + 8   (b) 17 + 21 + 39 + 13   (c) 15 + 7 + 243

But to compute two- or more digit numbers like (b) and (c) above, you have to rewrite the numbers vertically, one under the other. Since our number system is a place value system, **it is very important that you line up the digits correctly according to their place value.** Let's use (c) as an example.

$$\begin{array}{rrrr} & & 1 & 5 \\ & & & 7 \\ + & 2 & 4 & 3 \\ \hline \end{array}$$

Since 15, 7, 243, each has different number of digits, digits must be lined up correctly so that you will **add the digits with the same place value.** For example:

- 5, 7, 3, are lined up in the **ones** column;
- 1, 4, are lined up in the **tens** column;
- 2 alone in the **hundreds** column.

Do you know what would happen, if you **carelessly** write 7 under 1 (tens place) instead of 5 (ones place)? 7 becomes 70, **10 times larger.** keep place value in mind when you write numbers.

General Rules For Adding Numbers

You can compute ***any*** addition problem, large and small, **with confidence, if:**

1. **You have mastered the addition facts.** **(See p.131)**
2. **You understand the place value concept.** **(See p.18)**
3. **You have learned the skill of carrying.** **(See p.140)**

The digits must be lined up correctly.

Remember to ***add*** the number that was carried over.

Adding **always begins at the ones place**, then the tens place,...

$$\begin{array}{r} {}^{1} \\ 58 \\ +\ 25 \\ \hline 83 \end{array}$$

You can **add down** (8 + 5) or **add up** (5 + 8).

Write the answer under the line. Make sure **each digit is in the right place.**

Always check your answer. If added down, check by adding up or vice versa.

Addition - Carrying More Than 10 (Read first p.140)

The **speed** and **accuracy** with which you add **depends on your knowledge of the basic addition facts.** Remember, accuracy comes first; speed second.

$$\begin{array}{r} \overset{2}{3}8 \\ 27 \\ +\ 59 \\ \hline 4 \end{array}$$

Step 1. Add ones' place.

* Add: 8 + 7 + 9 = 24 (24 = 20 + 4).
* Write 4 in the ones place under 9.
Carry 20 to the tens place by writing 2 above 3.

$$\begin{array}{r} \overset{2}{3}8 \\ 27 \\ +\ 59 \\ \hline 124 \end{array}$$

Step 2. Add tens' place.

* Add: *2* + 3 + 2 + 5 = 12 (12 means 100 + 20).
(2, above 3, is carried over from the ones place.)
* Write 2 in the tens place under 5 and 1 at the left of 2, since there is no hundreds to add to.

To Check, reverse the order (add up):
* Add ones' place: 9 + 7 + 8 = 24 ✓
* Add tens' place: 2 + 3 + 2 + 5 = 12. ✓

Remember: When adding three or more addends, you could carry more than 10 from the ones' place to the tens' place.

Addition - Carrying More Than One Place (Read first p.140)

$$\begin{array}{r} 2\overset{1}{7}9 \\ +\ 486 \\ \hline 5 \end{array}$$

Step 1. Add ones' place.

* Add: 9 + 6 = 15. (15 = 10 + 5)

* Write 5 in the ones place. *Carry 1 to the tens place.*

$$\begin{array}{r} \overset{1}{2}\overset{1}{7}9 \\ +\ 486 \\ \hline 65 \end{array}$$

Step 2. Add tens' place.

* Add: *1* + 7 + 8 = 16 (16 means 100 + 60)

(1, above 7, was carried over from the ones place.)

* Write 6 in the tens place.

Carry 1 to the hundreds place.

$$\begin{array}{r} \overset{1}{2}79 \\ +\ 486 \\ \hline 765 \end{array}$$

Step 3. Add hundreds' place.

* Add: *1* + 2 + 4 = 7 (7 means 700)

(1, above 2, was carried over from the tens place.)

* Write 7 in the hundreds place under 4.

To Check, reverse the order (add up).

* Add ones' place: 6 + 9 = 15 ✓

* Add tens' place: 8 + 7 + 1 = 16 ✓

* Add hundreds place: 4 + 2 + 1 = 7 ✓

Summary

(Addition)

- Addition facts are used to compute all addition problems. These facts are also used in multiplication. Every math student should memorize them.
- From the addition table, you can find the sums of all one-digit additions, the addition properties, and different pairs of numbers that add up to the same number.
- To simplify computation, look for numbers, two or three, that add up to 10. Use what you already know to speed up your work.
- Writing numbers in expanded form can help us understand the process of adding two- or more digit numbers.
- The decimal system makes "carrying" in addition necessary At any place (ones, tens,), if the sum is 10 or more, carry the 10 to the next higher value place. Sometimes, you may have to carry 20 or more.
- To add two-or more digit numbers, write the numbers in vertical form with the value of each digit lined up correctly Then, start adding the numbers from the ones place, then the tens place, Always in that order Remember to add the numbers that are being carried over. Answer must also be lined up correctly according to their place value.
- Making it a habit of checking your answer If you added down, check by adding up. Addition is commutative.

the family **MATH.** companion

ARITHMETIC—THE FOUNDATION OF MATH

Ruth C. Sun

Part II. Whole Number Operations

B. Addition

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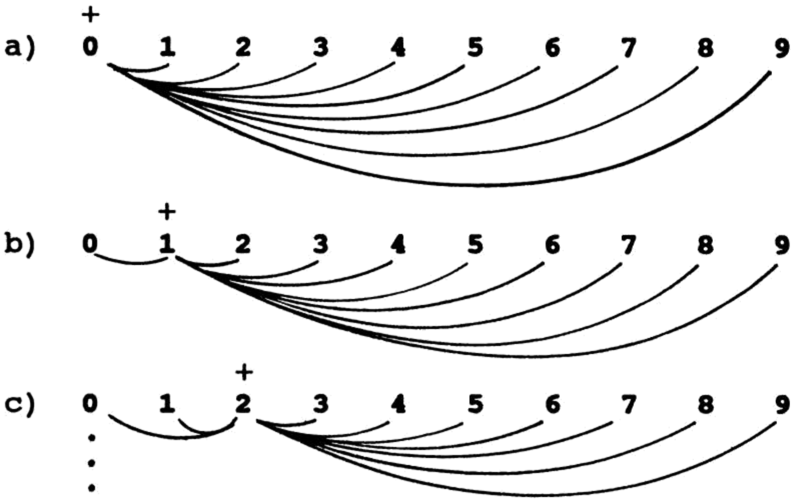
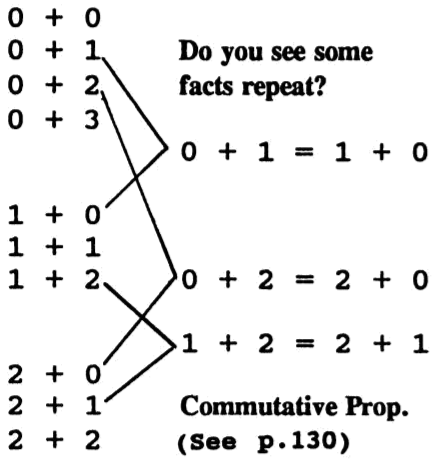
Addition Facts: Combinations of Ten Digits (See "Ten Digits" p.106)

The following shows how we got our addition facts. They are the combinations of the ten digits (0 to 9). Start with 0. Add 0 to itself, then added to each of the remaining digits, one at a time in order, as seen in a). Follow the same procedure for each digit, 1 through 9, and you will have the 100 basic addition facts found on the next page.




0 + 0
0 + 1
0 + 2
0 + 3

1 + 0
1 + 1
1 + 2

2 + 0
2 + 1
2 + 2

Do you see some facts repeat?

0 + 1 = 1 + 0

0 + 2 = 2 + 0

1 + 2 = 2 + 1

Commutative Prop. (See p.130)

Note: The sign "+" serves to remind you that you always add the number to itself.

100 Basic Addition Facts - One-Digit Addition

0 + 0 = 0
0 + 1 = 1
0 + 2 = 2
0 + 3 = 3
0 + 4 = 4
0 + 5 = 5
0 + 6 = 6
0 + 7 = 7
0 + 8 = 8
0 + 9 = 9

1 + 0 = 1
1 + 1 = 2
1 + 2 = 3
1 + 3 = 4
1 + 4 = 5
1 + 5 = 6
1 + 6 = 7
1 + 7 = 8
1 + 8 = 9
1 + 9 = 10

2 + 0 = 2
2 + 1 = 3
2 + 2 = 4
2 + 3 = 5
2 + 4 = 6
2 + 5 = 7
2 + 6 = 8
2 + 7 = 9
2 + 8 = 10
2 + 9 = 11

3 + 0 = 3
3 + 1 = 4
3 + 2 = 5
3 + 3 = 6
3 + 4 = 7
3 + 5 = 8
3 + 6 = 9
3 + 7 = 10
3 + 8 = 11
3 + 9 = 12

4 + 0 = 4
4 + 1 = 5
4 + 2 = 6
4 + 3 = 7
4 + 4 = 8
4 + 5 = 9
4 + 6 = 10
4 + 7 = 11
4 + 8 = 12
4 + 9 = 13

5 + 0 = 5
5 + 1 = 6
5 + 2 = 7
5 + 3 = 8
5 + 4 = 9
5 + 5 = 10
5 + 6 = 11
5 + 7 = 12
5 + 8 = 13
5 + 9 = 14

6 + 0 = 6
6 + 1 = 7
6 + 2 = 8
6 + 3 = 9
6 + 4 = 10
6 + 5 = 11
6 + 6 = 12
6 + 7 = 13
6 + 8 = 14
6 + 9 = 15

7 + 0 = 7
7 + 1 = 8
7 + 2 = 9
7 + 3 = 10
7 + 4 = 11
7 + 5 = 12
7 + 6 = 13
7 + 7 = 14
7 + 8 = 15
7 + 9 = 16

8 + 0 = 8
8 + 1 = 9
8 + 2 = 10
8 + 3 = 11
8 + 4 = 12
8 + 5 = 13
8 + 6 = 14
8 + 7 = 15
8 + 8 = 16
8 + 9 = 17

9 + 0 = 9
9 + 1 = 10
9 + 2 = 11
9 + 3 = 12
9 + 4 = 13
9 + 5 = 14
9 + 6 = 15
9 + 7 = 16
9 + 8 = 17
9 + 9 = 18

Addition Properties & Addition Facts (Review first "Properties" p.120)

It is **very** important that you memorize the addition facts because all future addition problems and other operations, depend on these basic facts. The following shows that by using **two addition properties,** you can cut down the memory work by half.

0 + 1 = 1 or 1 + 0 = 1
0 + 2 = 2 2 + 0 = 2
0 + 3 = 3 3 + 0 = 3
0 + 4 = 4 4 + 0 = 4
0 + 5 = 5 5 + 0 = 5

The **zero property** of addition says, **"Any number plus 0 is the number."** If you know this property, you can omit 19 facts from the list - they are the ones inside the boxes. (See p.129)

0 + 1 = 1 + 0
3 + 5 = 5 + 3
4 + 7 = 7 + 4
6 + 8 = 8 + 6
9 + 2 = 2 + 9

The **Commutative property** of addition says, **"The order in which numbers are added does not affect the result."** Again, by using this property, we cut down the numbers of facts to be memorized by half.
(See next page.)

Remember: These two addition properties apply also to other addition problems including decimals and fractions.

Rearranging The Addition Facts

By using the **zero property** and the **commutative property** of addition, the numbers of addition facts to be learned are reduced to 45 as listed below. In practicing, **make it a habit** of saying each addition facts **both ways** before giving the sum like this: **3 + 5 = 5 + 3 = 8** or **3 + 5 equals 5 + 3 is 8.**

* 1 + 1 = 2
1 + 2 = 2 + 1 = 3
1 + 3 = 3 + 1 = 4
1 + 4 = 4 + 1 = 5
1 + 5 = 5 + 1 = 6
1 + 6 = 6 + 1 = 7
1 + 7 = 7 + 1 = 8
1 + 8 = 8 + 1 = 9
1 + 9 = 9 + 1 = 10
* 2 + 2 = 4
2 + 3 = 3 + 2 = 5
2 + 4 = 4 + 2 = 6
2 + 5 = 5 + 2 = 7
2 + 6 = 6 + 2 = 8
2 + 7 = 7 + 2 = 9
2 + 8 = 8 + 2 = 10
2 + 9 = 9 + 2 = 11
* 3 + 3 = 6
3 + 4 = 4 + 3 = 7
3 + 5 = 5 + 3 = 8
3 + 6 = 6 + 3 = 9
3 + 7 = 7 + 3 = 10
3 + 8 = 8 + 3 = 11
3 + 9 = 9 + 3 = 12
* 4 + 4 = 8
4 + 5 = 5 + 4 = 9
4 + 6 = 6 + 4 = 10
4 + 7 = 7 + 4 = 11
4 + 8 = 8 + 4 = 12
4 + 9 = 9 + 4 = 13
* 5 + 5 = 10
5 + 6 = 6 + 5 = 11
5 + 7 = 7 + 5 = 12
5 + 8 = 8 + 5 = 13
5 + 9 = 9 + 5 = 14
* 6 + 6 = 12
6 + 7 = 7 + 6 = 13
6 + 8 = 8 + 6 = 14
6 + 9 = 9 + 6 = 15
* 7 + 7 = 14
7 + 8 = 8 + 7 = 15
7 + 9 = 9 + 7 = 16
* 8 + 8 = 16
8 + 9 = 9 + 8 = 17
* 9 + 9 = 18

Note: The addition facts with * have identical addends.

Addition Table

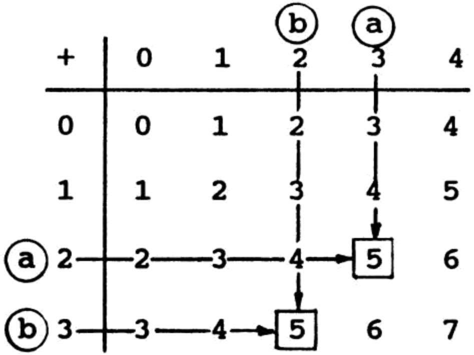
+	0	1	2	3	4	5	6	7	8	9 ← Addends
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

↑
Addends

Study the table carefully. Do you see the addition properties? Do you see different pairs of numbers that add up to the same number?

Using The Addition Table To Find Sums

In the addition table, the digits at the top and the digits at the left side are addends. There are **two ways** of looking up the sum of two numbers because ***additions are commutative***. See the example below:



+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7

Finding The Sum: 3 + 2 = 2 + 3 = □

- * Find 3 in the top row ⓐ , and 2 in the left column ⓐ . Then go down the column of 3 and go across the row of 2. **The sum is the number where the column and the row meet.** Or
- * Find 3 in the left column ⓑ , and 2 in the top row ⓑ . **The sum is where the column and the row meet.**

Patterns In Addition Table

Study carefully the addition table and see whether you can find the following:

- * Do you see all the addition properties?
- * Do you see the different pairs of numbers that give the same sum?

Numbers Adding Up To 10 (See also "Subtracting From 10" p.155)

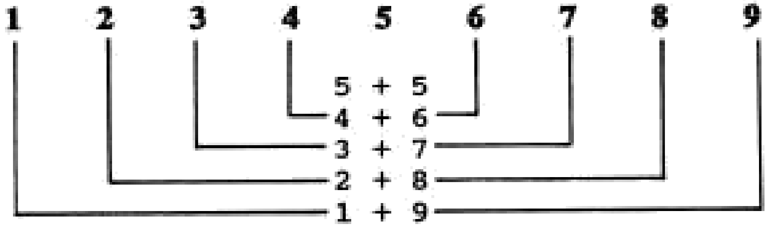
When adding a column of numbers, **always look for numbers, two or three, that add up to 10.** It helps to speed up the computations. See the examples below:

a) $\begin{array}{r} 4\\ 3\\ 6\\ +\ 7\\ \hline 20 \end{array}$ b) $\begin{array}{r} 7\\ 5\\ 2\\ +\ 1\\ \hline 15 \end{array}$ c) $\begin{array}{r} 16\\ 6\\ 3\\ +\ 4\\ \hline 29 \end{array}$ d) $\begin{array}{r} 1\\ 8\\ 9\\ +\ 2\\ \hline 20 \end{array}$

In a) 4 + 6 = 10; 3 + 7 = 10
b) 7 + 2 + 1 = 10
c) 16 + 4 = 20
d) 1 + 9 = 10; 8 + 2 = 10
Here we use the associative property

Use the following diagram to help you **memorize** the two numbers and three numbers that add up to ten.

Two numbers add up to 10:



Three numbers add up to 10:

1 + 1 + 8 = 10
1 + 2 + 7 = 10
1 + 3 + 6 = 10
1 + 4 + 5 = 10
2 + 2 + 6 = 10
2 + 3 + 5 = 10
2 + 4 + 4 = 10
3 + 3 + 4 = 10

Applying What You Know

Put your brain to work when studying mathematics! When doing math work, take a good look at the problem first. Always look for ways that you can apply what you already know **to simplify the computation.** For example, if you know the numbers that add up to 10, then you can compute the following problems mentally.

a) If you know: 2 + 8 = 10
Then apply: 12 + 8 = 20
32 + 18 = 50 ←

Remember, you can break numbers apart!
Think: 32 = 30 + 2; 18 = 10 + 8.
So, (32 + 8) + 10 = 50, or
30 + (2 + 18) = 50

b) If you know: 2 + 5 = 7
Then apply: 20 + 50 = 70
200 + 500 = 700

Check:

$$\begin{array}{r} 20 \\ +\ 50 \\ \hline 70 \end{array} \qquad \begin{array}{r} 200 \\ +\ 500 \\ \hline 700 \end{array}$$

If the addends have equal numbers of zeros at the end, add the non-zero digits and keep the number of zeros.

c) If you know: 4 + 6 = 10
Then apply: 40 + 60 = 100
400 + 600 = 1000

Check:

$$\begin{array}{r} 40 \\ +\ 60 \\ \hline 100 \end{array} \qquad \begin{array}{r} 400 \\ +\ 600 \\ \hline 1000 \end{array}$$

Playing With Numbers (I) - Breaking Numbers Apart

In the process of learning the basic facts, many students tend to see these facts as something fixed. In fact, **any digit or number larger than 1 can be broken apart into a sum of two smaller numbers.** If you know the **rules of mathematics,** you can manipulate numbers with ease. For example:

a) 2 + 5 = 7.   **Remember 7 can also be written as: 0 + 7, 1 + 6, 3 + 4.**

2 + 5 = 3 + ☐   **Replace 7 with 3 + 4.** Then cover 4 and ask your friend to **find the missing addend.** ***Fun!***

2 + 5 = 5 + ☐   How about **replace 7 with 5 + 2.** Cover 2 and ask your friend to **name the property!** ***Fun!***

b) 8 + 5 = ☐   You can break either 8 or 5 apart into two numbers, so there will be **two numbers that add up to 10.** ***Speed up!***

8 + 5 = ☐   **Think:** 8 = **3 + 5.**   So, 3 + **(5 + 5)** = 3 + **10** = 13
8 + 5 = ☐   **Think:** 5 = **2 + 3.**   So, **(8 + 2)** + 3 = **10** + 3 = 13

Rule: **You always compute the numbers inside the parentheses first.**
So, put the numbers that add to 10 inside the parentheses.

c) $13 - 4 = \square$

Break 13 apart. **subtract 4 from 10.** Then add the difference to the ones digit. (See p.155)

$10 + 3 - 4 =$

Think: 13 = 10 + 3.

$\underbrace{(10 - 4)}_{\downarrow} + 3 =$

Rearrange the numbers: Subtract 4 from 10.

$6 + 3 = 9$

Add the difference to 3. *Easy!*

d) $5 \times 27 = \square$

If you know that you can break numbers apart, you can solve the problem in your head.

$5 \times (20 + 7) =$

Think: 27 = **20 + 7,** because 5 x 20 is easy.

$\underset{\downarrow}{(5 \times 20)} + \underset{\downarrow}{(5 \times 7)} =$

Distributive property over addition.

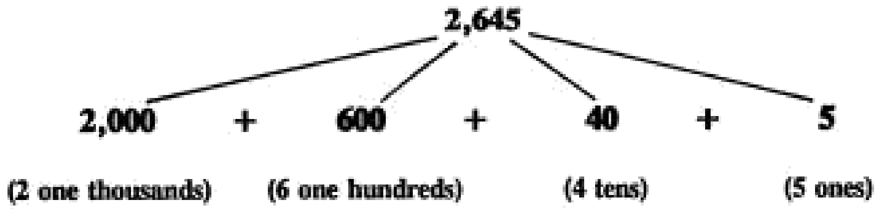
$100 + 35 = 135$

Multiply 5 x 27, you will get 135.

The examples given above are only a sample of what you can do with numbers. If you know numbers can be taken apart, you can speed up computations and turn many problems into **mental math.** *Math is fun!*

Writing Numbers in Expanded Form (See also "Expanded Form" p.24)

To write a number in expanded form is to write it **as the sum of the value of each digit.** The following shows 2645 in expanded form:



$$\begin{array}{r} 2,000 \\ 600 \\ 40 \\ +\quad 5 \\ \hline 2,645 \end{array}$$

A number can be expressed in various forms. We can write 2645

a) **in words:** "two thousands, six hundred forty-five"

b) **in standard numeral:** 2,645 (or 2645)

d) **in expanded form:** 2,645 = 2,000 + 600 + 40 + 5

It is important that you learn to read and write numbers correctly, especially large numbers. You can find the instructions on pages 18-19.

Addition Using Expanded Form (Review first the previous page.)

Writing addition problems in expanded form can help us to see the process of addition. First, write each addend in an expanded form.

Example: Add. 85 + 67

$$\begin{array}{rcl} 8\;5 & = & 80 + 5 \\ \underline{+\;6\;2} & = & \underline{\;60 + 2} \\ 7 & & 140 + 7 = 147 \\ \underline{+1\;4\;0} & & \\ 1\;4\;7 & & \end{array}$$

Writing each addend in expanded form helps us to see, that when we add **8 + 6 = 14,** we are actually adding **80 + 60 = 140** with **0 in the ones place omitted.** Do you know Why?

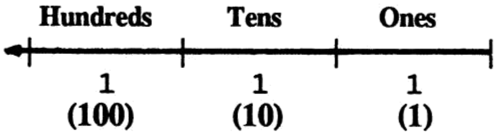
Example: Add. 8 + 97 + 103

$$\begin{array}{rcr} {}'\;\;{}'\;\;\; & & \\ 8 & & 8 \\ 9\;7 & = & 90 + \;\;7 \\ \underline{+1\;0\;3} & = & \underline{100 + 00 + \;\;3} \\ 2\;0\;8 & & 100 + 90 + 18 = 208 \end{array}$$

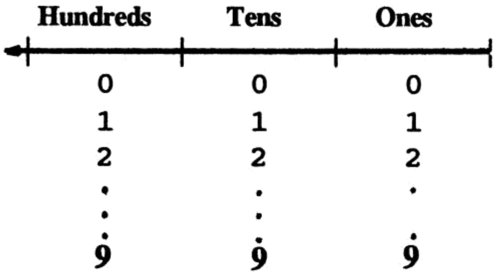
Make sure the place value of the digits are **lined up correctly.**
Then,

- Add the ones: 8 + 7 + 3 = 18. ***Carry*** **one 10 to the tens place by adding 1 to the tens.**
- Add the tens: 10 + 90 = 100. ***Carry*** **one 100 to the hundreds place.**

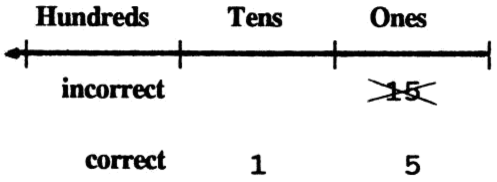
Carrying In Addition & Decimal System



Remember that in decimal system or base 10 system the value of each place is **10 times larger** than the place to its right.



According to base 10 system, each place (ones, tens, hundreds, etc.) can have ***only*** **one-digit numbers: 0, 1, 2, 3, ...** ***up to 9.*** That means we ***can not*** have **two-digit numbers** like 10, 11, 12..., etc. in any place. See the example below. Because of this, we have **"carrying"** in addition.



Therefore, in the process of adding numbers, **if the sum of any place (ones, tens, hundreds, etc.) is 10 or more,** we have to ***carry*** the 10, 20, etc. ***to the next higher place on the left.***

(Continued on the next page.)

Examples Of Carrying In Addition (First, read the previous page.)

(a)

$$\begin{array}{rr} & 4 \\ + & 5 \\ \hline & 9 \end{array}$$

(b)

$$\begin{array}{rrr} & & 2 \\ + & & 8 \\ \hline & 1 & 0 \end{array}$$

(c)

$$\begin{array}{rrr} & 1 & \\ & & 7 \\ + & 2 & 4 \\ \hline & 3 & 1 \end{array}$$

(d)

$$\begin{array}{rrr} & 2 & \\ & & 9 \\ & & 6 \\ + & 4 & 8 \\ \hline & 6 & 3 \end{array}$$

(e)

$$\begin{array}{rrrr} & & 5 & 5 \\ + & & 9 & 2 \\ \hline & 1 & 4 & 7 \end{array}$$

(a) Adding *ones* place: 4 + 5 = 9. *No carrying.*

(b) Adding *ones* place: 2 + 8 = 10. *Carrying.*
10 ones = 1 ten. So, write 1 in the tens place & 0 in the ones place.

(c) Adding *ones* place: 7 + 4 = 11. **Carrying.**
11 = 10 + 1 Carry 10 ones to the tens place by adding 1 (10 + 20 = 30)

(d) Adding *ones* place: 9 + 6 + 8 = 23. *Carrying.*
23 = 20 + 3. Carry 20 ones to the tens place by adding 2. (20 + 40 = 60)

(e) Adding *tens* place: 5 + 9 = 14. *Carrying.*
14 = 10 + 4. Carry 10 tens to hundreds place by adding 1

Note: We use the word "carrying" only when the place we carry to has numbers like (c) & (d).

Writing Numbers In Vertical Form **(Read first "Place Value" p.18)**

Math problems are often written horizontally to save space like the following examples:

(a) 2 + 4 + 6 + 8   (b) 17 + 21 + 39 + 13   (c) 15 + 7 + 243

But to compute two- or more digit numbers like (b) and (c) above, you have to rewrite the numbers vertically, one under the other. Since our number system is a place value system, **it is very important that you line up the digits correctly according to their place value.** Let's use (c) as an example.

$$\begin{array}{rrrr} & & 1 & 5 \\ & & & 7 \\ + & 2 & 4 & 3 \\ \hline \end{array}$$

Since 15, 7, 243, each has different number of digits, digits must be lined up correctly so that you will **add the digits with the same place value.** For example:

- 5, 7, 3, are lined up in the **ones** column;
- 1, 4, are lined up in the **tens** column;
- 2 alone in the **hundreds** column.

Do you know what would happen, if you **carelessly** write 7 under 1 (tens place) instead of 5 (ones place)? 7 becomes 70, **10 times larger.** keep place value in mind when you write numbers.

General Rules For Adding Numbers

You can compute *any* addition problem, large and small, **with confidence, if:**

1. **You have mastered the addition facts.** **(See p.131)**
2. **You understand the place value concept.** **(See p.18)**
3. **You have learned the skill of carrying.** **(See p.140)**

The digits must be lined up correctly.

Remember to ***add*** the number that was carried over.

Adding **always begins at the ones place**, then the tens place,...

$$\begin{array}{r} \overset{\prime}{5}8 \\ +\ 25 \\ \hline 83 \end{array}$$

You can **add down** (8 + 5) or **add up** (5 + 8).

Write the answer under the line. Make sure **each digit is in the right place.**

Always check your answer. If added down, check by adding up or vice versa.

Addition - Carrying More Than 10 **(Read first p.140)**

The **speed** and **accuracy** with which you add **depends on your knowledge of the basic addition facts.** Remember, accuracy comes first; speed second.

$$\begin{array}{r} \overset{2}{3}8 \\ 27 \\ +\ 59 \\ \hline 4 \end{array}$$

Step 1. Add ones' place.

* Add: 8 + 7 + 9 = 24 (24 = 20 + 4).

* Write 4 in the ones place under 9.
Carry 20 to the tens place by writing 2 above 3.

$$\begin{array}{r} \overset{2}{3}8 \\ 27 \\ +\ 59 \\ \hline 124 \end{array}$$

Step 2. Add tens' place.

* Add: *2* + 3 + 2 + 5 = 12 (12 means 100 + 20).
(2, above 3, is carried over from the ones place.)

* Write 2 in the tens place under 5 and 1 at the left of 2, since there is no hundreds to add to.

To Check, reverse the order (add up):

* Add ones' place: 9 + 7 + 8 = 24 ✓

* Add tens' place: 2 + 3 + 2 + 5 = 12. ✓

Remember: When adding three or more addends, you could carry more than 10 from the ones' place to the tens' place.

Addition - Carrying More Than One Place (Read first p.140)

$$\begin{array}{r} 2\overset{1}{7}9 \\ +\ 486 \\ \hline 5 \end{array}$$

Step 1. Add ones' place.

* Add: 9 + 6 = 15.   (15 = 10 + 5)

* Write 5 in the ones place. *Carry 1 to the tens place.*

$$\begin{array}{r} \overset{1}{2}\overset{1}{7}9 \\ +\ 486 \\ \hline 65 \end{array}$$

Step 2. Add tens' place.

* Add: *1* + 7 + 8 = 16   (16 means 100 + 60)

(1, above 7, was carried over from the ones place.)

* Write 6 in the tens place.

Carry 1 to the hundreds place.

$$\begin{array}{r} \overset{1}{2}79 \\ +\ 486 \\ \hline 765 \end{array}$$

Step 3. Add hundreds' place.

* Add: *1* + 2 + 4 = 7   (7 means 700)

(1, above 2, was carried over from the tens place.)

* Write 7 in the hundreds place under 4.

To Check, reverse the order (add up).

* Add ones' place: 6 + 9 = 15 ✓

* Add tens' place: 8 + 7 + 1 = 16 ✓

* Add hundreds place: 4 + 2 + 1 = 7 ✓

Summary
(Addition)

- Addition facts are used to compute all addition problems. These facts are also used in multiplication. Every math student should memorize them.
- From the addition table, you can find the sums of all one-digit additions, the addition properties, and different pairs of numbers that add up to the same number.
- To simplify computation, look for numbers, two or three, that add up to 10. Use what you already know to speed up your work.
- Writing numbers in expanded form can help us understand the process of adding two- or more digit numbers.
- The decimal system makes "carrying" in addition necessary At any place (ones, tens,), if the sum is 10 or more, carry the 10 to the next higher value place. Sometimes, you may have to carry 20 or more.
- To add two-or more digit numbers, write the numbers in vertical form with the value of each digit lined up correctly Then, start adding the numbers from the ones place, then the tens place, Always in that order Remember to add the numbers that are being carried over. Answer must also be lined up correctly according to their place value.
- Making it a habit of checking your answer If you added down, check by adding up. Addition is commutative.

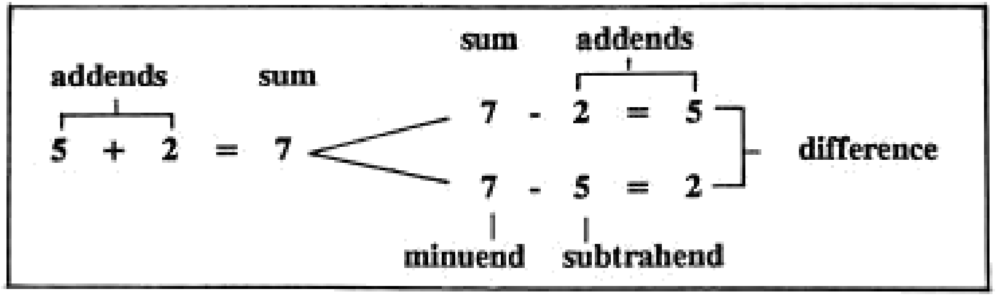
Part II. Whole Number Operations

C. Subtraction

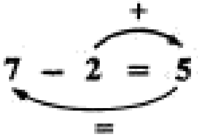
Table of Contents

Subtraction - An Inverse Operation of Addition (Review first p.110)

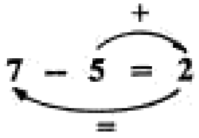
Subtraction is the **opposite** of addition. Instead of "adding on," we are "taking away." **For every addition fact, there are two subtraction facts:**



Inverse Operations - The following examples show what we mean when we say **subtraction is an inverse operation of addition** - an operation in reversed order:



(2 + 5 = 7) and



(5 + 2 = 7)

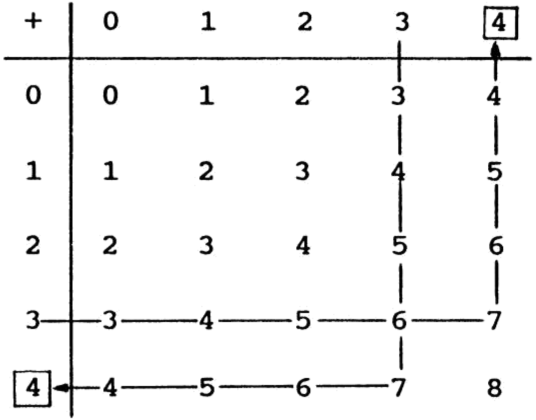
Remember: ***Subtractions are not commutative.*** Review the special rule for subtracting whole numbers found on page 122.

Using The Addition Table To Find Differences (See "Addition Table" p.132)

Again, the following example shows how subtraction is related to addition. **To find the difference is the same as to find the missing addend:**

$7 - 3 = \square$ **In subtraction,** we are finding the difference.
$3 + \square = 7$ **In addition,** we are finding the missing addend.

Just as in addition, there are **two ways** of finding the missing addend in the addition table as seen below:



+	0	1	2	3	$\boxed{4}$
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
$\boxed{4}$	4	5	6	7	8

Finding The Missing Addend: $3 + \square = 7$

- * Find 3 **in the top** row and go down the column, **stop at 7.** Then go across the row **to the left. The missing addend is the number at the end of the row.**
- * Or find 3 **at the left** side column and go across the row, **stop at 7.** Then **go straight up** to the top. **The missing addend is the number at the top.**

The missing addend is the difference.

100 Basic Subtraction Facts

0 − 0 = 0
1 − 0 = 1
2 − 0 = 2
3 − 0 = 3
4 − 0 = 4
5 − 0 = 5
6 − 0 = 6
7 − 0 = 7
8 − 0 = 8
9 − 0 = 9

1 − 1 = 0
2 − 1 = 1
3 − 1 = 2
4 − 1 = 3
5 − 1 = 4
6 − 1 = 5
7 − 1 = 6
8 − 1 = 7
9 − 1 = 8
10 − 1 = 9

2 − 2 = 0
3 − 2 = 1
4 − 2 = 2
5 − 2 = 3
6 − 2 = 4
7 − 2 = 5
8 − 2 = 6
9 − 2 = 7
10 − 2 = 8
11 − 2 = 9

3 − 3 = 0
4 − 3 = 1
5 − 3 = 2
6 − 3 = 3
7 − 3 = 4
8 − 3 = 5
9 − 3 = 6
10 − 3 = 7
11 − 3 = 8
12 − 3 = 9

4 − 4 = 0
5 − 4 = 1
6 − 4 = 2
7 − 4 = 3
8 − 4 = 4
9 − 4 = 5
10 − 4 = 6
11 − 4 = 7
12 − 4 = 8
13 − 4 = 9

5 − 5 = 0
6 − 5 = 1
7 − 5 = 2
8 − 5 = 3
9 − 5 = 4
10 − 5 = 5
11 − 5 = 6
12 − 5 = 7
13 − 5 = 8
14 − 5 = 9

6 − 6 = 0
7 − 6 = 1
8 − 6 = 2
9 − 6 = 3
10 − 6 = 4
11 − 6 = 5
12 − 6 = 6
13 − 6 = 7
14 − 6 = 8
15 − 6 = 9

7 − 7 = 0
8 − 7 = 1
9 − 7 = 2
10 − 7 = 3
11 − 7 = 4
12 − 7 = 5
13 − 7 = 6
14 − 7 = 7
15 − 7 = 8
16 − 7 = 9

8 − 8 = 0
9 − 8 = 1
10 − 8 = 2
11 − 8 = 3
12 − 8 = 4
13 − 8 = 5
14 − 8 = 6
15 − 8 = 7
16 − 8 = 8
17 − 8 = 9

9 − 9 = 0
10 − 9 = 1
11 − 9 = 2
12 − 9 = 3
13 − 9 = 4
14 − 9 = 5
15 − 9 = 6
16 − 9 = 7
17 − 9 = 8
18 − 9 = 9

Subtraction Properties & Subtraction Facts (Review "Properties" p.120)

Like the addition facts, **the subtraction facts must also be memorized!** You can use **two subtraction properties** to reduce the number of subtraction facts to be mastered.

$0 - 0 = 0$
$1 - 0 = 1$
$2 - 0 = 2$
$3 - 0 = 3$

The **identity property of subtraction** says, **"A number minus zero is the number."** Knowing this property helps you to omit **10 facts** from the list. They are the facts found inside the box (p.152)

$\boldsymbol{n - 0 = n}$

Generalization: *n* can be ***any*** number - 99 or 278.

$1 - 1 = 0$
$2 - 2 = 0$
$3 - 3 = 0$
$4 - 4 = 0$

The **zero property of subtraction** says, **"A number minus itself equals 0."** If you know this property, you can omit **another 9 facts** - the facts inside the box on page 152.

$\boldsymbol{m - m = 0}$

Generalization: *m* can be ***any*** number - 35 or 999.

Remember: Mathematical properties generally have wide applications. Knowing them now will help you in your study of math in the days to come.

Rearranging the Subtraction Facts

[2 - 1 = 1]

* 3 - 1 = 2
3 - 2 = 1

* 4 - 1 = 3
4 - 3 = 1
[4 - 2 = 2]

* 5 - 1 = 4
5 - 4 = 1
* 5 - 2 = 3
5 - 3 = 2

* 6 - 1 = 5
6 - 5 = 1
* 6 - 2 = 4
6 - 4 = 2
[6 - 3 = 3]

* 7 - 1 = 6
7 - 6 = 1
* 7 - 2 = 5
7 - 5 = 2
* 7 - 3 = 4
7 - 4 = 3

* 8 - 1 = 7
8 - 7 = 1
* 8 - 2 = 6
8 - 6 = 2
* 8 - 3 = 5
8 - 5 = 3
[8 - 4 = 4]

* 9 - 1 = 8
9 - 8 = 1
* 9 - 2 = 7
9 - 7 = 2

* 9 - 3 = 6
9 - 6 = 3
* 9 - 4 = 5
9 - 5 = 4

* 10 - 1 = 9
10 - 9 = 1
* 10 - 2 = 8
10 - 8 = 2
* 10 - 3 = 7
10 - 7 = 3
* 10 - 4 = 6
10 - 6 = 4
[10 - 5 = 5]

* 11 - 2 = 9
11 - 9 = 2
* 11 - 3 = 8
11 - 8 = 3

* 11 - 4 = 7
11 - 7 = 4
* 11 - 5 = 6
11 - 6 = 5

* 12 - 3 = 9
12 - 9 = 3
* 12 - 4 = 8
12 - 8 = 4
* 12 - 5 = 7
12 - 7 = 5
[12 - 6 = 6]

* 13 - 4 = 9
13 - 9 = 4
* 13 - 5 = 8
13 - 8 = 5
* 13 - 6 = 7
13 - 7 = 6

* 14 - 5 = 9
14 - 9 = 5
* 14 - 6 = 8
14 - 8 = 6
[14 - 7 = 7]

* 15 - 6 = 9
15 - 9 = 6
* 15 - 7 = 8
15 - 8 = 7

* 16 - 7 = 9
16 - 9 = 7
[16 - 8 = 8]

* 17 - 8 = 9
17 - 9 = 8

[18 - 9 = 9]

You may find it easier to learn each pair of subtraction facts together (*). **Each pair is related to one addition facts.** (See p.150)

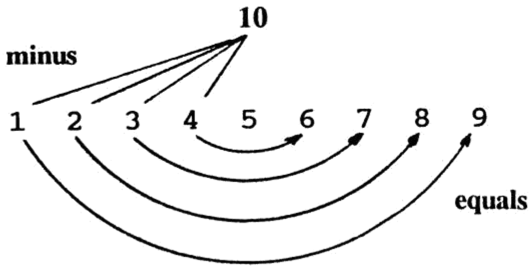
Subtraction Model 1 - Subtracting from 10 (Review first "Place Value" p.18)

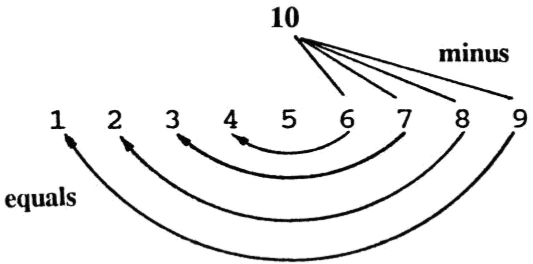
Many students have difficulties learning the subtraction facts found inside the boundary (previous page) because **the subtrahend is larger than the minuend in the ones place.** If you are one of them, try one of the following methods.

Method 1. Subtracting From 10:

$13 - 5$	**Think:** 13 = 10 + 3
$(10 - 5) + 3$	Subtract 5 from 10 instead of 13.
$5 + 3 = 8$	Then add the difference to 3.

Learn to subtract from 10: (See also "Add Up To 10" p 134)





Subtraction Model 2 - Adding to 10 (See also "Playing With Numbers" pp.136-137)

$$\begin{array}{ccc} \begin{array}{r} 3 \\ -\,2 \\ \hline 1 \end{array} & \begin{array}{c} (3 + 2) \\ (2 + 2) \\ \end{array} & \begin{array}{r} 5 \\ -\,4 \\ \hline 1 \end{array} \\ & = & \end{array}$$

The example shows, **adding the same number to *both*** minuend and subtrahend **does not change** the difference.

Model 2. Adding to 10:

$$17 \quad - \quad 9$$

Since subtracting 10 is easy, change 9 to 10.

$$\begin{array}{ccccc} (17 + 1) & - & (9 + 1) & & \\ \downarrow & & \downarrow & & \\ 18 & - & 10 & = & 8 \end{array}$$

Add 1 to *both* minuend and subtrahend.

Then subtract 10 from 18.

Note: This method is useful if the subtrahend **is close to 10,** such as 8 or 9.

Remember: Each subtraction model is trying to manipulate numbers to make it easier to subtract. So, **make sure you learn it right and do it correctly!**

Subtraction Model 3 - Subtracting Twice

Model 3. Subtracting Twice:

15 - 8	**Think: 5 – 8. 5 is not enough. 5 needs 3 more to make it 8.**
(8 - 5 = 3) (5 + 3 = 8)	**Actually, you are finding the difference between 8 and 5. Or You remember 5 + 3 = 8.**
10 - 3 = 7	**Then you take the 3 needed from 10.**

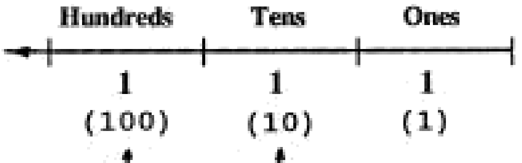
Comparing Model 1 and 3:
Both models deal with subtraction in which the subtrahend is larger than the minuend. But here is **the difference between the two:**

* **In Model 1:** **You start at tens place.** You subtract from 10, then add the difference to the number in the ones place.

* **In Model 3:** **You start at the ones place.** You take from the number in the ones place, then take what is short of from 10.

Borrowing In Subtraction & Decimal System (Review "Carrying" p.140)

To understand the concept of borrowing, you must first understand our number system. According to **base-10 system,** if the sum of any place is 10 or more, you ***carry*** the 10 to the next higher place by adding 1; if carry 20, add 2, etc. ***Borrowing*** **is just the opposite of carrying** as seen below:



Remember that number in any place is **10 times larger** than the number ***to its right.***

If you ***borrow*** **1 from the tens place,** you get **10 ones.**
(In ***carrying***, 10 ones becomes 1 ten.)

If you ***borrow*** **1 from the hundreds place,** you get **10 tens** or
9 tens and 10 ones (9 tens and 10 ones = 10 tens = 100)
(In ***carrying,*** 10 tens becomes 1 hundred.)

Remember, ***in borrowing,*** **we do not take away anything from the number.** **We just break the number apart, and regroup the number to make the subtraction possible.**

Borrowing In Subtraction - Using Expanded Form (Review "Expanded Form" p.24)

Students, in general, find borrowing harder to do than carrying. In carrying, you just add the numbers. But in borrowing, you have to **regroup numbers** which may involve more than one place.

(a) $$\begin{array}{r} 3\overset{\downarrow}{4} \\ \underline{-\ 9} \end{array}$$

(b) $$\begin{array}{r} 3\overset{\downarrow}{1}9 \\ \underline{-\ 64} \end{array}$$

Borrowing becomes necessary **when you try to subtract a larger number from a smaller number.** For example:

- In (a), it happens in the *ones place.*
- In (b), it happens in the *tens place.*

If we write the numbers in expanded form, it would help us to see the process of borrowing. Let's use the example (a) above.

Standard Method **Expanded Method**

$$\begin{array}{rcrcrcl}
 & & \ulcorner\!\longrightarrow & \longrightarrow & 30 = 20 + 10 & & \\
 & & \mid\qquad & & \downarrow \quad \swarrow & & \\
\overset{2}{\cancel{3}}\,\overset{14}{4} & & 30 + 4 & & 20 + 14 & & \\
\underline{-\quad 9} & = & \underline{-\qquad 9} & = & \underline{-\qquad 9} & & \\
2\;5 & & & & 20 + 5 & = & 25
\end{array}$$

Since 4 - 9 is impossible, we have to ***borrow*** **1 from the tens place.** That means, we **borrow 10 from 30 to make 14 in the ones place.**

Borrowing Across Zeros - Using Expanded Form

Example: Subtract. 502 - 7

a) 502 - 7

500 + 00 + 2

b) 400 + 100 + 2

c) 400 + 90 + (10 + 2)

502 - 7 = 495

Since 2 - 7 is impossible, we have to ***borrow*** **one 10 from the tens place.** Let's use expanded form to see how it works.

Since the tens-place is *empty*, we have to ***borrow*** **a 100 from the hundreds place** to fill the tens place.

Next, we ***borrow*** **a 10 from the tens place** and **add to the 2 in the ones place** to make it 12. Now, we can subtract 7 from 12.

Shortcut:

* To subtract across zero like 502 - 7, think of ***borrowing*** ***1 from 50,*** and change 50 to 49 with 12 in the ones place: $\overset{4}{\mathbf{5}}\,\overset{9}{\mathbf{0}}\,\overset{12}{\mathbf{2}}$

* To subtract across zeros like 3005 - 8, think of ***borrowing*** ***1 from 300,*** and change 300 to 299 with 15 in the ones place: $\overset{2}{\mathbf{3}}\,\overset{9}{\mathbf{0}}\,\overset{9}{\mathbf{0}}\,\overset{15}{\mathbf{5}}$

General Rules For Subtracting Numbers

**The larger number is always at the top.
The digits must be lined up correctly.**

At any place, if the minuend is smaller than subtrahend, borrow 1 from the number in the next higher place. Then reduce that number by 1.

$$\begin{array}{rr} \overset{3}{4} & \overset{17}{7} \\ -\,1 & 9 \\ \hline 2 & 8 \end{array}$$

Begin at the right - subtract the ones, then the tens,.... until every place has been subtracted.

To check: Add the difference to the subtrahend. The sum should equal the minuend.

Remember: No matter how large the number may be, you subtract **only two numbers at a time** (See p.123). And each time you use the subtraction facts to compute. The fact is, you **can do any** subtraction problem with confidence, **if you have mastered two things:**

1. **Basic subtraction facts** (See p.154)
2. **The skill of borrowing** (See p.158)

Subtraction - Borrowing Twice (Review first p.158)

The **speed** and **accuracy** with which you subtract depend on your knowledge of the basic subtraction facts and the skill of borrowing. Remember, accuracy always comes first! Example: Subtract. 625 - 147.

$$\begin{array}{r} 6\overset{1}{\cancel{2}}\overset{15}{5} \\ -\ 147 \\ \hline 8 \end{array}$$

Step 1. Subtract the ones: 5 - 7 = impossible.

* Borrow 1 from the tens place: **2 became 1.**
Strike out 2, write 1 above it.

* 15 - 7 = 8. Write 8 in the ones place under 7.

$$\begin{array}{r} \overset{5}{\cancel{6}}\overset{11}{2}5 \\ -\ 147 \\ \hline 78 \end{array}$$

Step 2. Subtract the tens: 1 - 4 = impossible.

* Borrow 1 from the hundreds place: **6 became 5.**
Strike out 6, write 5 above it.

* 11 - 4 = 7. Write 7 in the tens place under 4.

$$\begin{array}{r} \overset{5}{\cancel{6}}25 \\ -\ 147 \\ \hline 478 \end{array}$$

Step 3. Subtract the hundreds.

* 5 - 1 = 4. Write 4 in the hundreds place under 1.

Check: The difference plus subtrahend should equal minuend.

(difference) + (subtrahend) = (minuend)
478 + 147 = 625 ✓

Subtracting Across Zeros (See also p.160)

Subtract. 500 - 316. Remember, **"0s" are place holders** which means the ones and the tens places are **open** for regrouping.

$$\begin{array}{cccc} & 4 & 9 & 10 \\ & \cancel{5} & \cancel{0} & 0 \\ - & 3 & 1 & 6 \\ \hline \end{array}$$

Step 1. Regrouping the tens and ones places.

* First, borrow 1 from the hundreds place, **5 becomes 4,** and the **1 borrowed becomes 10** in the tens-place.
* Again, borrow 1 from the tens place, **10 becomes 9,** and the **1 borrowed becomes 10** in the ones place.

$$\begin{array}{cccc} & 4 & 9 & 10 \\ & \cancel{5} & \cancel{0} & 0 \\ - & 3 & 1 & 6 \\ \hline & 1 & 8 & 4 \end{array}$$

Step 2. Subtract.

* Subtract ones place: 10 - 6 = 4. Write 4 in the ones place under 6.
* Subtract tens place: 9 - 1 = 8 Write 8 in the tens place under 1.
* Subtract hundreds place: 4 - 3 = 1. Write 1 in the hundreds place under 3.

Check: 184 + 316 = 500. ✓

Shortcut: To borrow across zeros like 500, think of borrowing 1 from 50 and **change 50 to 49** with **10 in the ones place.**

Summary
(Subtraction)

- Subtraction and addition are inverse operations. There are two subtraction facts for every addition fact.
- We can use the addition table to find the difference. To find the difference is the same as to find the missing addend.
- We can always add but we can't always subtract because subtraction is not commutative. If we subtract a larger number from a smaller number, the answer will be a "negative" number
- Subtraction facts are important. We use these facts to subtract large and small numbers; only two numbers at a time. These facts are also used in division.
- Borrowing is the opposite of carrying. In borrowing, we regroup the number to make the subtraction possible.
- In subtracting numbers, line up the numbers correctly in vertical form with the larger number at the top. Then, subtract the ones, followed by the tens, always in that order At any place when the minuend is smaller than subtrahend, borrow 1 from the number in the next larger place and reduce that number by 1
- To check, add the difference to the subtrahend. The sum should equal the minuend.

Part II. Whole Number Operations

D. Multiplication

Table of Contents

Multiplication - A Repeated Addition (See p.110)

Suppose you bought a box of chocolate which had 6 rows with 8 pieces in each row. If you want to know **how many pieces of chocolate are altogether**, you can use one of the following methods to find out:

By Adding:

$$\underbrace{8 + 8 + 8 + 8 + 8 + 8}_{\textbf{6 addends}\text{ (6 rows)}} = \underbrace{48}_{\textbf{sum}\text{ (total)}}$$

By Multiplying:

$$\underbrace{8}_{\substack{\textbf{multiplicand}\\ \text{(8 pieces)}}} \times \underbrace{6}_{\substack{\textbf{multiplier}\\ \text{(6 rows)}}} = \underbrace{48}_{\substack{\textbf{product}\\ \text{(total)}}}$$

Read: "8 times 6 equals 48" or "6 eights equal 48."

It's **much faster to use multiplication** when we have to **add the same number many times.**

Factors & Multiples (Read also pp.282-283)

Factors are numbers that you multiply together to get products. Therefore, factors are another name for **multiplicand** and **multiplier.** **Multiple** is another name for product but with this difference:

factor (multiplicand)		**factor** (multiplier)		**product**
3	x	1	=	3
3	x	2	=	6
3	x	3	=	9
3	x	4	=	12
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

multiples of 3

The difference between product and multiple:

* **3, 6, 9, 12, ..., *each* is a product of two factors.**

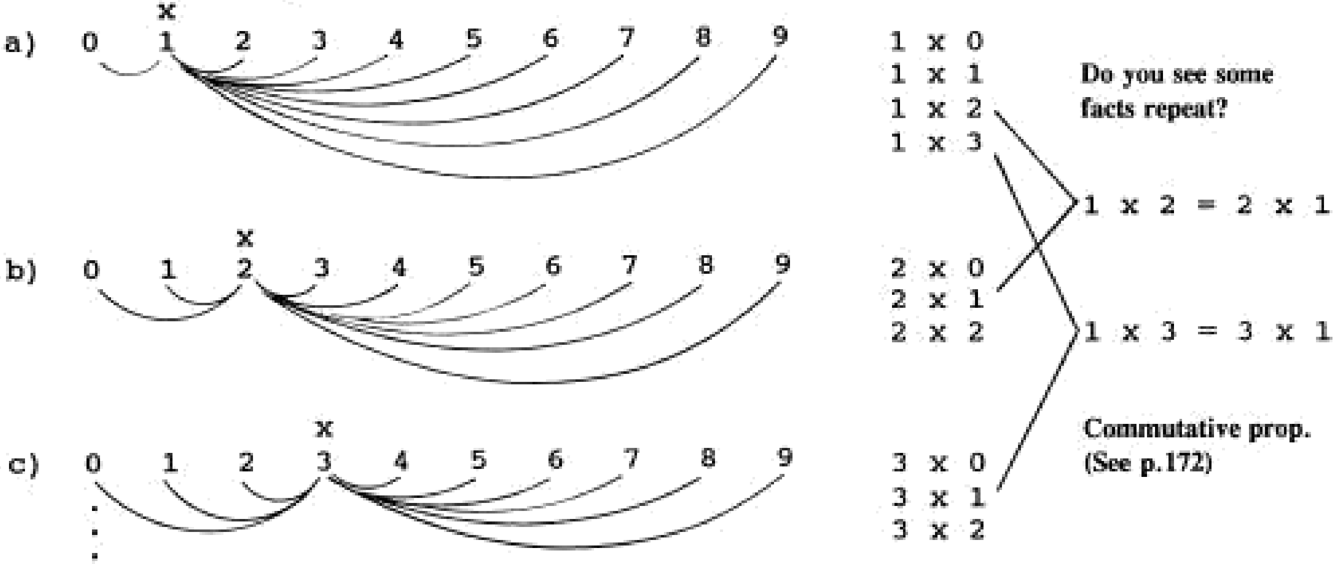
Factors come in pairs (two) because we multiply only two numbers at a time.

* **3, 6, 9, 12, ..., are *all* multiples of 3.**

Connection: Factors and multiples are important concepts in fractions.

Multiplication Facts: Combinations of 10 Digits (See also p.128)

The multiplication facts, like the addition facts, are the combinations of the ten digits (0, 1, 2,..., 9). **Skip 0** because 0 times any number is 0. **Start with 1. Multiply 1 to each digit** ***including itself*** like a). Follow the same procedure for each digit, and you will have the 90 multiplication facts listed on next page.



Do you see some facts repeat?

Commutative prop. (See p.172)

90 Basic Multiplication Facts -- One-Digit Multiplication

Any number times zero is zero!

1 x 0 = 0
1 x 1 = 1
1 x 2 = 2
1 x 3 = 3
1 x 4 = 4
1 x 5 = 5
1 x 6 = 6
1 x 7 = 7
1 x 8 = 8
1 x 9 = 9

2 x 0 = 0
2 x 1 = 2
2 x 2 = 4
2 x 3 = 6
2 x 4 = 8
2 x 5 = 10
2 x 6 = 12
2 x 7 = 14
2 x 8 = 16
2 x 9 = 18

3 x 0 = 0
3 x 1 = 3
3 x 2 = 6
3 x 3 = 9
3 x 4 = 12
3 x 5 = 15
3 x 6 = 18
3 x 7 = 21
3 x 8 = 24
3 x 9 = 27

4 x 0 = 0
4 x 1 = 4
4 x 2 = 8
4 x 3 = 12
4 x 4 = 16
4 x 5 = 20
4 x 6 = 24
4 x 7 = 28
4 x 8 = 32
4 x 9 = 36

5 x 0 = 0
5 x 1 = 5
5 x 2 = 10
5 x 3 = 15
5 x 4 = 20
5 x 5 = 25
5 x 6 = 30
5 x 7 = 35
5 x 8 = 40
5 x 9 = 45

6 x 0 = 0
6 x 1 = 6
6 x 2 = 12
6 x 3 = 18
6 x 4 = 24
6 x 5 = 30
6 x 6 = 36
6 x 7 = 42
6 x 8 = 48
6 x 9 = 54

7 x 0 = 0
7 x 1 = 7
7 x 2 = 14
7 x 3 = 21
7 x 4 = 28
7 x 5 = 35
7 x 6 = 42
7 x 7 = 49
7 x 8 = 56
7 x 9 = 63

8 x 0 = 0
8 x 1 = 8
8 x 2 = 16
8 x 3 = 24
8 x 4 = 32
8 x 5 = 40
8 x 6 = 48
8 x 7 = 56
8 x 8 = 64
8 x 9 = 72

9 x 0 = 0
9 x 1 = 9
9 x 2 = 18
9 x 3 = 27
9 x 4 = 36
9 x 5 = 45
9 x 6 = 54
9 x 7 = 63
9 x 8 = 72
9 x 9 = 81

Multiplication Properties & Multiplication Facts (Review "Properties" p.121)

Learn the mathematic properties, and use them to your advantage! For example, if you know the multiplication properties, you can cut the need for memorization in half:

$1 \times 0 = 0$ $2 \times 0 = 0$ $3 \times 0 = 0$	The **zero property of multiplication** says, **"A number times 0 is 0."** If you know the property, you don't have to memorize 9 of the facts.
$n \times 0 = 0$	**Generalization:** *n* can be any number -- 33, 128, etc.
$1 \times 1 = 1$ $1 \times 2 = 2$ $1 \times 3 = 3$	The **identity property of multiplication** says, **"Any number multiplied by 1 is the number."** Knowing this property, you can omit another 9 facts.
$1 \times n = n$	**Generalization:** *n* can be any number -- 99, 9,999, etc.
$2 \times 3 = 3 \times 2$ $2 \times 4 = 4 \times 2$ $3 \times 6 = 6 \times 3$	The **commutative property of multiplication** says, **"The order in** which you multiply the numbers **does not affect the answer."** Therefore, you have only 36 **facts** to remember.
$a \times b = b \times a$	**Generalization:** *a* and *b* can be any number, **except 0.**

Rearranging The Multiplication Facts

Attention Please! **Your ability to do division and fractions depend on your knowledge of multiplication facts.** Memorize them now so that you will enjoy working with division and fractions later. If you know the multiplication properties, all you need to memorize are the following facts:

* 2 x 2 = 4
2 x 3 = 3 x 2 = 6
2 x 4 = 4 x 2 = 8
2 x 5 = 5 x 2 = 10
2 x 6 = 6 x 2 = 12
2 x 7 = 7 x 2 = 14
2 x 8 = 8 x 2 = 16
2 x 9 = 9 x 2 = 18

* 5 x 5 = 25
5 x 6 = 6 x 5 = 30
5 x 7 = 7 x 5 = 35
5 x 8 = 8 x 5 = 40
5 x 9 = 9 x 5 = 45

* 3 x 3 = 9
3 x 4 = 4 x 3 = 12
3 x 5 = 5 x 3 = 15
3 x 6 = 6 x 3 = 18
3 x 7 = 7 x 3 = 21
3 x 8 = 8 x 3 = 24
3 x 9 = 9 x 3 = 27

* 6 x 6 = 36
6 x 7 = 7 x 6 = 42
6 x 8 = 8 x 6 = 48
6 x 9 = 9 x 6 = 54

* 7 x 7 = 49

* 4 x 4 = 16
4 x 5 = 5 x 4 = 20
4 x 6 = 6 x 4 = 24
4 x 7 = 7 x 4 = 28
4 x 8 = 8 x 4 = 32
4 x 9 = 9 x 4 = 36

7 x 8 = 8 x 7 = 56
7 x 9 = 9 x 7 = 63

* 8 x 8 = 64
8 x 9 = 9 x 8 = 72

* 9 x 9 = 81

Facts with "*" are squared numbers (See p.41)

Suggestion: In practicing, say facts **both ways** before giving the answer. It helps you to remember the commutative property.

Multiplication Table

– factors

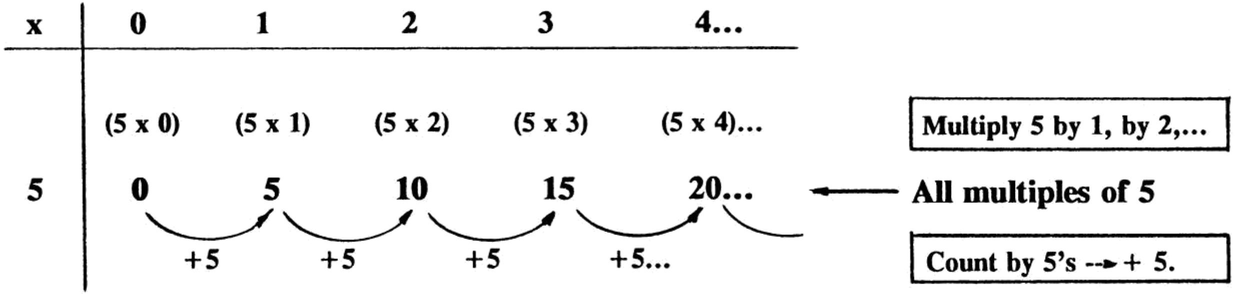
x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

factors

Study the table carefully Do you see the multiplication properties? Do you see different pairs of factors that give the same product?

Multiples: A Pattern In Multiplication Table (see also "Multiples" p.283)

In the multiplication table, **the numbers in each row are the multiples of the number heading the row;** and **the numbers in each column are the multiples of the number heading the column.** Let's take the row of 5 as an example:

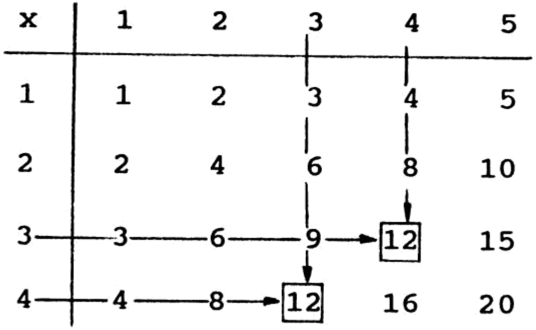


Two Ways of Finding the Multiples of A Number:

1. **Multiply the number by 1, by 2,..., or by any whole number.** **All** the products are the multiples of the number. **Or**
2. **Count by the number (skip count),** say **count by 5's:** 5, 10, 15, 20,...**(repeatedly add 5)**. They are multiples of 5.

Using The Multiplication Table To Find Products (Compare with p.133)

In the multiplication table, **the digits at the top** and **the digits at the left side are factors.** The rest of the numbers are the products of two factors. The procedure of finding the product of two numbers is the same as finding the sum of two addends. There are **two ways:**



x	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20

Finding The Product: $4 \times 3 = \square$

- * Find the factor 4 **in the top row,** and the factor 3 **in the left column.** Then go down the column of 4 and go across the row of 3. **The product is the number where the column and the row meet. Or**
- * Find 4 in the left column and 3 in the top row. The product is the number where the row and the column meet.

Observe The Pattern: (see p.174)

$$4 \times 3 = \begin{cases} (4 \times 2) + \square = 12 \\ (4 \times 4) - \square = 12 \end{cases}$$

Can you find the missing numbers? Do you see the pattern?

Playing With Numbers (II) (Read also p.136)

Since any number **larger than 1** can be written as **a product of two factors** or **a sum of two smaller numbers**, we can play with either factor. For example:

a) $4 \times 6 = 4 \times \underline{3 \times 2} = 24$

Think: 4 x 6 is the **double** of 4 x 3, because **6 = 3 x 2.**

$4 \times 6 = 4 \times 3 \times \square = 24$

How about removing 2 and ask your friend to find the missing number.

b) $4 \times 6 = 4 \times (5 + 1) = 24$

$= 4 \times 5 + 4 \times 1 = 24$

Write **6 as (5 + 1), using a parenthesis.** Then use **distributive property.** (See p.121)

$4 \times 6 = 4 \times (5 + \square) = 24$

How about removing 1 and ask your friend to find the missing number.

c) $8 \times 9 = (8 \times 10) - 8 = 72$

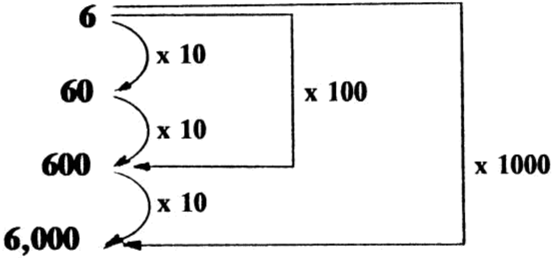
$= 8 \times (10 - 1)$

$= \underline{8 \times 10} - \underline{8 \times 1} = 72$

Since **8 x 9 + 8** = 8 x 10, we can write 8 x 9 as **8 x 10 - 8.**
Or write **9 as (10 - 1)** and then use **distributive property.**
Either way gives the same answer.

Multiplying by 10, 100, 1000, ... (See also p.67)

Since our number system is based on 10, multiplying a number by 10, 100, 1000, ... is very easy to work with. ***Memorize the rules given below!*** The rule is very useful in multiplication/division of whole numbers and decimals; also in scientific notation.



Remember: Any number ending with zero(s) has a factor of 10, 100, 1000,...

60 = 6 x 10
600 = 6 x 100
6,000 = 6 x 1000

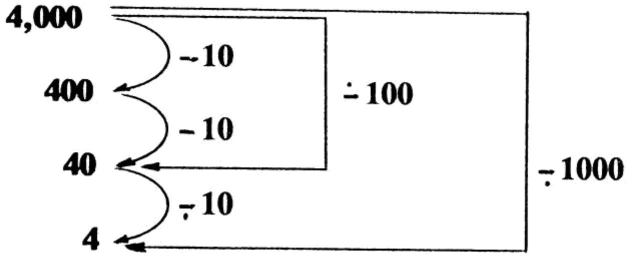
Rule:

To multiply by 10, 100, 1000, etc., add the number of zeros in the multiplier to the multiplicand:

- * multiply by 10, add 1 zero. *(Add 1 zero = multiply by 10.)*
- * multiply by 100, add 2 zeros. *(Add 2 zeros = multiply by 100.)*
- * multiply by 1000, add 3 zeros. *(Add 3 zeros = multiply by 1000.)*

Dividing By 10, 100, 1000, ... (Compare with "Multiplying By 10, 100,...)

This page is put here for the purpose of comparing with the opposite page. Dividing a number which ends in zeros by 10, 100, etc. is the opposite of multiplying a number by 10, 100, ... Either operation can be done mentally.



Writing the division in fraction also help us to understand:

$$\frac{4000}{10} = \frac{400 \times \cancel{10}}{\cancel{10}} = 400$$

$$\frac{40\cancel{00}}{1\cancel{00}} = 40$$

Rule:

To divide a number which ends in zeros by 10, 100, etc., cancel the same number of zeros in both the divisor and the dividend:

* Divide by 10, cancel 1 zero. *(Drop 1 zero = divide by 10)*
* divide by 100, cancel 2 zeros. *(Drop 2 zeros = divide by 100)*

Multiplying By Multiples of 10, 100,... (See also "Multiples" p.283)

Multiples of 10 are: 10, 20, 30, 40,... (Multiply 10 by 1, by 2,...)

Multiples of 100 are: 100, 200, 300,... (multiply 100 by 1, by 2,...)

a) $25 \times 30 = 25 \times \underline{3 \times 10} = 750$ **Multiply 25 x 3, then add 1 zero.**

b) $43 \times 200 = 43 \times \underline{2 \times 100} = 8600$ **Multiply 43 x 2, then add 2 zeros.**

Multiplying Numbers Ending With Zeros

c)

$$\begin{array}{rl} 3\,0 & - (3 \times 10) \\ \times\ 5 \vdots 0 & - (5 \times 10) \\ \hline 1\,5\,0\,0 & - (15 \times 100) \end{array}$$

1st. Multiply 3 x 5 = 15.
2nd. Add the **total number of zeros** found in **two factors (2 zeros)** to 15.

d)

$$\begin{array}{rl} 1\,5\,0 & - (15 \times 10) \\ \times\ 5 \vdots 0\,0 & - (5 \times 100) \\ \hline 7\,5\,0\,0\,0 & - (75 \times 1000) \end{array}$$

1st. Multiply 15 x 5 = 75.
2nd. Add the **total number** of zeros in **two factors (3 zeros)** to 75.

Prior Knowledge For Multiplying Numbers (Review first "Place-Value" p.109)

If you answer **"yes"** to the following questions without hesitation, then you can do any multiplication, large or small, with confidence.

1. **Can you give multiplication facts quickly from memory with no error?** (p.173)
2. **Do you understand the concept of place value?** (p.109)
3. **Have you mastered the skill of carrying?** (p.140)
4. **Have you mastered the skill of addition?** (p.143)

Understand The Vocabulary of Multiplication:

2 1 3		**multiplicand** (factor)
x 2 1		**multiplier** (factor)
2 1 3	←	***partial product*** (213 x 1 = 213)
+ 4 2 6	←	***partial product*** (213 x 2 = 426)
4 4 7 3		**product** (213 + 426 = 4473)

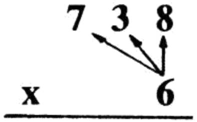
The partial products occur when the multiplier is two digits or more. Each time when we multiply the multiplicand by a digit of the multiplier, we get a partial product. Therefore, if multiplied by:

* a **two-digits** multiplier - we have **two partial products.**
* a **three-digits** multiplier - we have **three partial products.**

General Process Of Multiplying Numbers

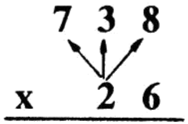
We multiply the top number (multiplicand) by **each digit** of the bottom number (multiplier) starting from the ones digit as you see below. Example: Multiply 738 x 26

Step 1.



Step 1. 738 x 6 (1st digit of multiplier)
Begin with the *ones* digit of the multiplicand:
1st. 6 x 8 = 48
2nd. 6 x 3 = 18 (actually 6 x 30 = 180)
3rd. 6 x 7 = 42 (actually 6 x 700 = 42,000)
Remember always to add the carried over numbers.

Step 2.



Step 2. 738 x 2 (2nd digit of multiplier)
Again, begin with the *ones* digit of the multiplicand:
1st. 2 x 8 = 16 (actually 20 x 8 = 160)
2nd. 2 x 3 = 6 (actually 20 x 30 = 600)
3rd. 2 x 7 = 14 (actually 20 x 700 = 14,000)

Note: We multiply larger numbers ***in exactly the same way*** we multiply smaller numbers. The larger numbers have more digits, that's all. If it were a three-digit multiplier, you continue the process by multiplying the top number by the 3rd digit of the multiplier, and so on.

General Rule For Placing The Partial Products (Review the previous pages first)

Since our number system is a **place value system,** it is important that each partial product be placed in a proper place. Let's use 1357 x abc for an illustration:

$$\begin{array}{ccccccccl}
 & & & & 1 & 3 & 5 & 7 & \\
 & & & \times & & c & b & a & \\ \cline{4-8}
 & & & & a & a & a & a & ① \\
 & & & b & b & b & b & & ② \\
 & + & c & c & c & c & & & ③ \\ \hline
 & & p & p & p & p & p & p & ④
\end{array}$$

Study "Multiplying by 10, 100,..." p.178 and "Multiplying By Three-Digits" p.186.

Step 1. Multiply 1357 by a ***(ones).***
Write the 1st partial product under the line **with the first digit in the** ***ones*** **place.** ①

Step 2. Multiply 1357 by b ***(tens)***
Write the 2nd partial product under the 1st one **with the first digit** ***in the same column with b*** **in the multiplier -** ***tens*** place. ②

Step 3. Multiply 1357 by c ***(hundreds)***
Write the 3rd partial product under the 2nd one **with the first digit** ***in the same column with*** **c in the multiplier -** ***hundreds*** place. ③

The product of 1357 x abc is ***the sum*** **of all the partial products.** ④ ***Make sure all the digits are lined up correctly*** before adding.

Multiplying By One-Digit Numbers With Carrying (Review first pp.182-183)

$$\begin{array}{cccc} & 2 & 4 & \\ & 7 & 3 & 8 \\ \times & & & 6 \\ \hline 4 & 4 & 2 & 8 \end{array}$$

1st. Multiply 6 x 8 *(ones)* = 48. (48 = 40 + 8).
Write 8 in the ones place under 6.
Carry 4 (40) to the tens place. Write 4 above 3.

2nd. Multiply 6 x 3 *(tens)* = 18. (18 means 100 + 80)
Add 18 + 4 carried = 22. (22 means 200 + 20)
Write 2 in the tens place at the left of 8.
Carry 2 (200) to the hundreds place. Write 2 above 7.

3rd. Multiply 6 x 7 *(hundreds)* = 42. (42 means 4000 + 200)
Add 42 + 2 carried = 44. (44 means 4000 + 400)
Since there is no digit in the thousands place, write 44 (means 4400) to the left of 2.

Do you notice that we used one-digit multiplication facts in each step? **The fact is you will be using these basic facts in doing all multiplication/division of whole numbers, decimals, and fractions.** Memorize the facts now, if you haven't done so!

Multiplication Without Carrying - In case of multiplication without carrying, write the product of each place (ones, tens,...) under that very same place. (ones, tens,...).

Multiplying By Two-Digit Numbers With Carrying **(Review first pp.182-183)**

```
      6 9 7     Same as       697          697
  x     3 2                 x  30        x   2
    1 3 9 4                 20910   +    1394   =   22304
+ 2 0 9 1 0
  2 2 3 0 4                Step 2       Step 1
```

Step 1. Multiply 697 by 2 *(ones)*. (See the previous page.)
Step 2. Multiply 697 by 3 *(tens)*. (Remember, 3 in the tens place = 30)

1st. Multiply 3 x 7 *(ones)* = 21. (actually 30 x 7 = 210)
Write 1 (means 10) in the ***tens*** place under 9. (Omit 0 in ones place.)
Carry 2 (means 200) **to the hundreds place.** (See next page.)

2nd. Multiply 3 x 9 *(tens)* = 27. (actually 30 x 90 = 2700)
Add 27 + 2 carried = 29. (actually 29 means 2000 + 900)
Write 9 in the hundreds place under 3, at the left of 1.
Carry 2 (means 2000) **to the thousands place.**

3rd. Multiply 3 x 6 *(hundreds)* = 18. (actually 30 x 600 = 18,000)
Add 18 + 2 carried = 20. (20 here means 20,000)
Write 20 at the left of 9.

Multiplying By Two-Digit Numbers With Carrying (Review first pp.182-183)

```
      6 9 7     Same as      697        697
  x     3 2                x  30      x   2
    1 3 9 4                20910  +   1394   =   22304
+ 2 0 9 1 0
  2 2 3 0 4                Step 2     Step 1
```

Step 1. Multiply 697 by 2 *(ones).* (See the previous page.)

Step 2. Multiply 697 by 3 *(tens).* (Remember, 3 in the tens place = 30)

1st. Multiply 3 x 7 *(ones)* = 21. (actually 30 x 7 = 210)

Write 1 (means 10) in the ***tens*** place under 9. (Omit 0 in ones place.)

Carry 2 (means 200) **to the hundreds place.** (See next page.)

2nd. Multiply 3 x 9 *(tens)* = 27. (actually 30 x 90 = 2700)

Add 27 + 2 carried = 29. (actually 29 means 2000 + 900)

Write 9 in the hundreds place under 3, at the left of 1.

Carry 2 (means 2000) **to the thousands place.**

3rd. Multiply 3 x 6 *(hundreds)* = 18. (actually 30 x 600 = 18,000)

Add 18 + 2 carried = 20. (20 here means 20,000)

Write 20 at the left of 9.

Multiplying By Three-Digit Numbers (Review first the last two pages.)

```
          5 4 2
        x 4 3 2
        -------
        1 0 8 4   = 542 x 2          First partial product    (Step 1)
      1 6 2 6     = 542 x 3  (30)    Second partial product   (Step 2)
  + 2 1 6 8       = 542 x 4  (400)   Third partial product    (Step 3)
  -------------
    2 3 4 1 4 4                      The product
```

Step 1. Multiply 542 by 2 ***(ones).*** (See "Multiplying By One-Digit")
542 x 2 = 1084. Write 1084, the first partial product, under the multiplier below the line with ***4 in the ones place under 2.***

Step 2. Multiply 542 by 3 ***(tens).*** (See "Multiplying By Two-Digit")
542 x 3 = 1626. Write 1626 under the first partial product with ***6 in the same column (place) with 3 in the multiplier.***
Second partial product is actually 542 x 30 = 16260 with 0 in the ones place omitted because 4 + 0 = 4. (Adding zeros does not change the sum.)

Step 3. Multiply 542 by 4 ***(hundreds).*** Follow same procedure.
542 x 4 = 2168. Write 2168 under the second partial product with ***8 in the same column (place) with 4 in multiplier***
Third partial product is actually 542 x 400 = 216800 with 0s in ones and tens places omitted.

Checking Multiplication (Review first "Checking Multiplication" p.115)

Making it a habit, always check your answers. You know by now, when comes to mathematics, the answer is ***either correct or incorrect.*** Mathematics is an exact science which knows no mercy! If you want to get good marks on tests or homework, **go over your work and check the following:**

* **Multiplication**
  - Are multiplications done in order?
  - Are all one-digit multiplications correct?
* **Carrying**
  - Are there numbers needed to be carried over?
* **Addition**
  - Are numbers carried being added?
  - Are additions done correctly?
* **Place-Value**
  - Are the product of each digit written in the correct value-place?
  - Are the partial products lined up correctly?

Suggestion: At the beginning of your study of multiplication, you should use one of the following methods to check your answer:

1. **(multiplicand) x (multiplier)** –(change the place)→ **(multiplier) x (multiplicand)**

   Multiplication is commutative, the order of the factors does not affect the product.

2. **(Product) ÷ (multiplier) = (multiplicand)** or **(product) ÷ (multiplicand) = (multiplier)**

   Division and multiplication are inverse operations.

Summary

(Multiplication)

* Multiplication is a repeated addition. When adding the same number over more than once, multiplication is a faster way to compute.

* Multiplication facts are used not only in all multiplication problems, but also in divisions and fractions. Students should memorize the multiplication facts. Multiplication properties help us cut down the number of multiplication facts to be memorized.

* From the multiplication table, you can find the products of all one-digit multiplication facts, the multiplication properties, and different pairs of factors that give the same product.

* Powers of 10 is easy to work with. For example, to multiply a number by a power of 10, we just add to the multiplicand the exact number of zeros that are in the multiplier

* To multiply two-or more digit numbers, write the numbers vertically Then multiply the multiplicand, in turn, by each digit of the multiplier beginning at ones digit. Place each partial product in the proper place. The product is the sum of all partial products.

* To check multiplication, change the positions of the factors and multiply Multiplication is commutative.

Part II. Whole Number Operations

E. Division

Table of Contents

Division - A Repeated Subtraction (See also p.110)

If multiplication is a repeated addition, **division is a repeated subtraction.** Suppose you want to divide 48 pieces of candy among 12 students, you can use **either subtraction or division** to find out **how many pieces each student will get.**

Using Repeated Subtraction:

$48 - \overset{\textcircled{1}}{12} = 36$ $36 - \overset{\textcircled{2}}{12} = 24$ $24 - \overset{\textcircled{3}}{12} = 12$ $12 - \overset{\textcircled{4}}{12} = 0$

You keep taking 12 pieces away from 48, until there is nothing left. The number of times you subtract is the answer - 4 pieces. There are four 12 pieces in 48.

Using Division:

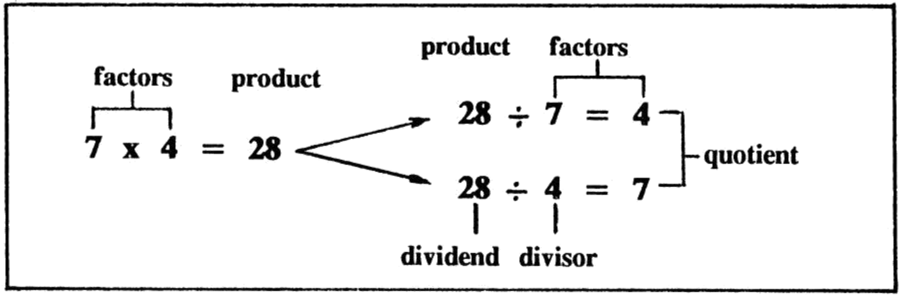
$48 - 12 = 4$ or $\frac{48}{12} = 4$

We are dividing 48, the total, into 12 equal parts, and we get 4 pieces.

Division is a much faster way to divide things/numbers into *equal parts*.

Division - An Inverse Operation of Multiplication (See also p.110)

Division is the **opposite** of multiplication. The following shows that for **every multiplication fact, except multiplication by zero, there are two related division facts:**



Although division and multiplication are inverse operations, **multiplication is commutative** ***while division is not.*** For example:

Multiplication: 12 x 2 = 24 **and** 2 x 12 = 24
Division: 12 ÷ 2 = 6 **but** 2 ÷ 12 = impossible

Therefore, in division **with** ***whole numbers,*** **the dividend must be equal to or larger than the divisor.** (See p.122)

90 Basic Division Facts - One-Digit Divisor

Division by zero
is impossible!

0 ÷ 1 = 0
1 ÷ 1 = 1
2 ÷ 1 = 2
3 ÷ 1 = 3
4 ÷ 1 = 4
5 ÷ 1 = 5
6 ÷ 1 = 6
7 ÷ 1 = 7
8 ÷ 1 = 8
9 ÷ 1 = 9

0 ÷ 2 = 0
2 ÷ 2 = 1
4 ÷ 2 = 2
6 ÷ 2 = 3
8 ÷ 2 = 4
10 ÷ 2 = 5
12 ÷ 2 = 6
14 ÷ 2 = 7
16 ÷ 2 = 8
18 ÷ 2 = 9

0 ÷ 3 = 0
3 ÷ 3 = 1
6 ÷ 3 = 2
9 ÷ 3 = 3
12 ÷ 3 = 4
15 ÷ 3 = 5
18 ÷ 3 = 6
21 ÷ 3 = 7
24 ÷ 3 = 8
27 ÷ 3 = 9

0 ÷ 4 = 0
4 ÷ 4 = 1
8 ÷ 4 = 2
12 ÷ 4 = 3
16 ÷ 4 = 4
20 ÷ 4 = 5
24 ÷ 4 = 6
28 ÷ 4 = 7
32 ÷ 4 = 8
36 ÷ 4 = 9

0 ÷ 5 = 0
5 ÷ 5 = 1
10 ÷ 5 = 2
15 ÷ 5 = 3
20 ÷ 5 = 4
25 ÷ 5 = 5
30 ÷ 5 = 6
35 ÷ 5 = 7
40 ÷ 5 = 8
45 ÷ 5 = 9

0 ÷ 6 = 0
6 ÷ 6 = 1
12 ÷ 6 = 2
18 ÷ 6 = 3
24 ÷ 6 = 4
30 ÷ 6 = 5
36 ÷ 6 = 6
42 ÷ 6 = 7
48 ÷ 6 = 8
54 ÷ 6 = 9

0 ÷ 7 = 0
7 ÷ 7 = 1
14 ÷ 7 = 2
21 ÷ 7 = 3
28 ÷ 7 = 4
35 ÷ 7 = 5
42 ÷ 7 = 6
49 ÷ 7 = 7
56 ÷ 7 = 8
63 ÷ 7 = 9

0 ÷ 8 = 0
8 ÷ 8 = 1
16 ÷ 8 = 2
24 ÷ 8 = 3
32 ÷ 8 = 4
40 ÷ 8 = 5
48 ÷ 8 = 6
56 ÷ 8 = 7
64 ÷ 8 = 8
72 ÷ 8 = 9

0 ÷ 9 = 0
9 ÷ 9 = 1
18 ÷ 9 = 2
27 ÷ 9 = 3
36 ÷ 9 = 4
45 ÷ 9 = 5
54 ÷ 9 = 6
63 ÷ 9 = 7
72 ÷ 9 = 8
81 ÷ 9 = 9

Division By Zero - Impossible! (See "Division Facts" p.194)

In studying mathematics, whenever you see words like "shortcut," "impossible," etc., **try, as much as you can, to understand the reason behind it.**

We have only 90 basic division facts instead of 100 because "division by 0 is impossible." Here is the reason why. Read carefully the following because it is **a logical reasoning.** Let's use $9 \div 0$ as an example:

Suppose we say	$9 \div 0 = n$	We use *n* to stand for the quotient.
Then	$n \times 0 = 9$	Because division and multiplication are **inverse operations** (See p.198).
The fact	$n \times 0 = 0$	We know **any number times zero is zero** ***not 9.*** (Zero property of multiplication)
Since	$n \times 0 \neq 9$	Which means **n x 0** ***can never*** **equal to 9.**
Therefore	$9 - 0$	*is impossible!*

Connection: We can not have ZERO as the denominator of a fraction, because division by zero is impossible.

Division Properties & Multiplication Properties (Review also p 121)

If you remember the following division properties, you do not have to memorize the facts found inside the boxes on page 194. Since division is an inverse operation of multiplication, their properties are related.

a) $0 \div n = 0$ * Zero property of division: **"Zero divided by any non-zero number is zero."** (*n* can be 1, 2, 3,. .)

$0 \times n = 0$ * Zero property of multiplication: **"Any number times zero is zero."** (*n* can be 0, 1, 2, 3,...)

b) $n \div 1 = n$ * Identity property of division: **"Any non-zero number divided by 1 is the number."** (*n* can be 1, 2,...)

$n \times 1 = n$ * Identity property of multiplication: **"Any number times 1 is the number."** (*n* can be 0, 1, 2,...)

c) $n - n = 1$ * One property of division: **"Any non-zero number divided by itself is 1."** (*n* can be 1, 2, 3,...)

$1 \times n = n$ * Identity property of multiplication: **"Any number times 1 is the number."** (*n* can be 0, 1, 2,...)

Note that both division properties, b) and c), came from the same multiplication property (p.172).

Rearranging Division Facts

[4 ÷ 2 = 2]
* 6 ÷ 2 = 3
6 ÷ 3 = 2
* 8 ÷ 2 = 4
8 ÷ 4 = 2
* 10 ÷ 2 = 5
10 ÷ 5 = 2
* 12 ÷ 2 = 6
12 ÷ 6 = 2
* 14 ÷ 2 = 7
14 ÷ 7 = 2
* 16 ÷ 2 = 8
16 ÷ 8 = 2
* 18 ÷ 2 = 9
18 ÷ 9 = 2

[36 ÷ 6 = 6]
* 42 ÷ 6 = 7
42 ÷ 7 = 6

[9 ÷ 3 = 3]
* 12 ÷ 3 = 4
12 ÷ 4 = 3
* 15 ÷ 3 = 5
15 ÷ 5 = 3
* 18 ÷ 3 = 6
18 ÷ 6 = 3
* 21 ÷ 3 = 7
21 ÷ 7 = 3
* 24 ÷ 3 = 8
24 ÷ 8 = 3
* 27 ÷ 3 = 9
27 ÷ 9 = 3

* 48 ÷ 6 = 8
48 ÷ 8 = 6
* 54 ÷ 6 = 9
54 ÷ 9 = 6

[16 ÷ 4 = 4]
* 20 ÷ 4 = 5
20 ÷ 5 = 4
* 24 ÷ 4 = 6
24 ÷ 6 = 4
* 28 ÷ 4 = 7
28 ÷ 7 = 4
* 32 ÷ 4 = 8
32 ÷ 8 = 4
* 36 ÷ 4 = 9
36 ÷ 9 = 4

[49 ÷ 7 = 7]
* 56 ÷ 7 = 8
56 ÷ 8 = 7
* 63 ÷ 7 = 9
63 ÷ 9 = 7

[25 ÷ 5 = 5]
* 30 ÷ 5 = 6
30 ÷ 6 = 5
* 35 ÷ 5 = 7
35 ÷ 7 = 5
* 40 ÷ 5 = 8
40 ÷ 8 = 5
* 45 ÷ 5 = 9
45 ÷ 9 = 5

[64 ÷ 8 = 8]
* 72 ÷ 8 = 9
72 ÷ 9 = 8

[81 ÷ 9 = 9]

Note: Each pair of the facts headed by "*" is related to a multiplication fact. (See p 173)

Remember: Your ability to do division depends on your knowledge of multiplication facts.

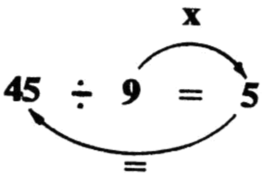
Three Ways Of Writing Division (see also "Symbols For Multiplication" p.113)

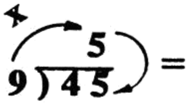
There are three ways of writing **"45 divided by 9 equals 5"** or **"9 into 45 is 5"** as seen below. ***Just Remember that*** **division can be written in fraction form.**

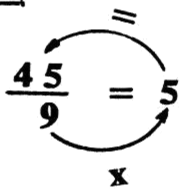
$$45 \div 9 = 5 \qquad \begin{array}{r} 5 \\ 9\overline{)45} \end{array} \qquad \frac{45}{9} = 5 \ (45/9 = 5)$$

Again, we can see that division and multiplication are inverse operations from the following demonstrations:

divisor (9)	x	**quotient** (5)	=	**dividend** (45)	
(factor)		(factor)		(product)	







Question: Do you see that the divisor and quotient are factors? Now, if the divisor is 5, what will be the quotient?

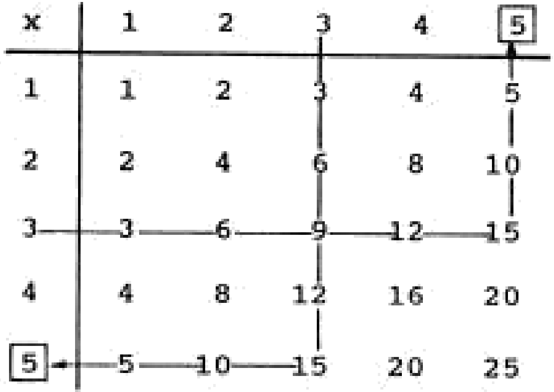
Using The Multiplication Table To Find Quotients (Review first p.176)

Since division is an inverse operation of multiplication, **finding the quotient is the same as finding the missing factor** as shown below:

15 ÷ 3 = ☐ **In division,** we are finding the quotient

3 x ☐ = 15 **In multiplication,** we are finding the missing factor.

There are **two ways** of finding a missing factor on the multiplication table because multiplication, like addition, is **commutative:**



x	1	2	3	4	$\boxed{5}$
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
$\boxed{5}$	5	10	15	20	25

Finding The Missing Factor: 3 x ☐ = 15

- * Find 3 in the top row and go down the column, stop at 15. Then go across the row to the left. **The missing factor is the number heading the row.**
- * Find 3 in the first column on the left. Go across the row, stop at 15. Then go straight up to the top. **The missing factor is the number heading the column.**

The missing factor is the quotient.

Writing Quotients With Remainders (Review p.96)

The following examples show that **not every whole number can be *divided evenly without remainder:***

a)

```
       9
 3 ) 27
   - 27
   ----
      0  ←—0 remainder
```

b)

```
       4
 6 ) 27
   - 24
   ----
      3  ←—3 remainder
      ↑
```

A remainder is a part of dividend that is left over
A remainder must always be less than the divisor: 3 < 6.

Three Ways of Writing A Remainder: (See p.251)

1) **As A Whole Number**

```
     4 r3
 6 ) 27
   - 24
   ----
      3
```

2) **As A Fraction** (p 96)

```
     4 1/2
 6 ) 27
   - 24
   ----
      3
```

3) **As A Decimal** (p.248)

```
      4.5
 6 ) 27.0
   - 24
   -----
      3 0
      3 0
      ---
        0
```

Interpreting The Remainder (Review previous page.)

In real life, remainders are not just numbers that we write in this way or that way. We must know what the remainder means, and how to interpret it correctly in a given situation. For example,

Problem: "Mary's class is planning for a class picnic. They are going to serve hot dogs with other goodies. There will be 33 students, the teacher, and two mothers at the picnic. For each one to have one hot dog, **how many packages** of hot dogs should they buy? Hot dogs come in a package of 8 links."

Solution: 1st. Decide **the numbers of people** who will attend the picnic:

33 (students) + **1** (teacher) + **2** (mothers) = **36**

2nd. Divide 36 by 8 (8 links in a package):

36 ÷ 8 = **4 with a remainder 4**

If they buy 4 packages of hot dogs, it will serve only 32 people: 4 (packages) x 8 (links) = 32
The remaining 4 people, r4, will go hungry.

Answer: For every one to have a hot dog, **they must buy 5 packages.**

Playing With Numbers (III) - Developing The Number Sense

In studying arithmetic, it is important to develop the number sense. Number sense is the **ability to see the relations that exist between numbers, and to use that existing relations to simplify the computation.** With practice, you will be able to tell which problem can be easily simplified and which one can not.

Example: Multiply 25 x 16. **(Remember: 25 x 16 = 16 x 25)**

$\boxed{4 \text{ x } 4 = 16}$

Method 1. **(25 x 4)** x 4

100 x 4 = 400

If you remember 25 x 4 = 100 and 16 = 4 x 4, you can solve the problem right in your head.

$\boxed{100 \div 4 = 25}$

Method 2. **(16 x 100)** ÷ 4

1600 ÷ 4 = 400

If you know that 25 is 1/4 of 100, you can multiply 16 by 100 and then divide the product by 4.

Example: Multiply 2.84 x 5. **(Remember: 5 = 10 ÷ 2)**

(2.84 x 10) ÷ 2

28.4 ÷ 2 = 14.2

You can multiply the number by 10, and then divide the product by 2.

Prior Knowledge For Dividing Numbers

Division is more involved than the other three operations. However, if you can answer **"yes" with confidence to each of the following questions,** you will have little difficulty doing division.

1. **Can you give one-digit multiplication facts quickly from memory with no error? (p.173)**
2. **Can you do subtraction with borrowing fast with no error? (pp.158-160)**
3. **Do you know how to estimate? (p.119)**
4. **Do you understand place-value concept? (p.109)**

Division and Multiplication Facts: Your ability to do division depends on your knowledge of multiplication facts because division uses multiplication facts in an opposite ways (See p 193). For example,

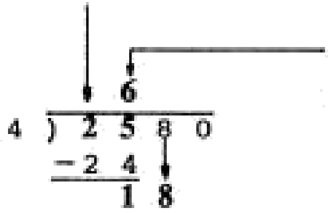
When you divide: 42 ÷ 6 (Inverse operations. p.198)
You think: "6 times *what* is 42?" or
"How many times 6 is contained in 42?"
You know: 6 x 7 = 42. So, 42 ÷ 6 = 7.

The Skill of Estimation: You use this skill -

* to estimate the **size of the quotient** - in the tens? in the hundreds?...
* to determine the **partial quotients** - using trial quotients.

General Rule For Dividing Numbers

Division begins at the left.



Partial quotients refers to the digits that make up the quotient. **Where to place the first digit** of the quotient is very important. See next page.

Partial dividends: In dividing, we divide only a part of the dividend at a time. **The dividend becomes a series of partial dividends.** Using the above example:

* **The 1st partial dividend is 25,** since 2 cannot be divided by 4:
(partial dividend) **25 ÷ 4 = 6** **(partial quotient)** ***remainder 1***

* **The 2nd partial dividend is 18,** the remainder and the next lower digit:
(partial dividend) **18 − 4 = 4** **(partial quotient)** ***remainder 2***

> **Rule: In division with *whole numbers,* the dividend must be equal to or larger than the divisor** (See p.122) Therefore, a partial dividend ***must have as many digits, or one digit more*** than the divisor.

Placing The First Partial Quotient (Review "Zeros Before..." p.66)

General rule: Place the first partial quotient right above *the right-hand digit* of the partial dividend, if the partial dividend has more than one digit. Consider each of the following dividends as if they were the first partial dividend of a division problem:

a)

$$\begin{array}{r} 0 \\ 4\overline{)\;2} \end{array}$$

2 can't be divided by 4. Write a zero above the digit 2. But zeros before whole numbers are omitted. (p.66)

b)

$$\begin{array}{r} 2 \\ 3\overline{)\;\mathbf{6}} \end{array}$$

6 divided by 3 is 2 (3 x 2 = 6). **Write 2 above 6.**

c)

$$\begin{array}{r} 8 \\ 7\overline{)\;5\;\mathbf{9}} \end{array}$$

56 divided by 8 is 8 r3. **Write 8 above 9, *the right-hand digit*** of the partial dividend 59

d)

$$\begin{array}{r} 4 \\ 1\;2\overline{)\;4\;\mathbf{9}} \end{array}$$

49 divided by 12 is 4 r1. **Write 4 above 9, *the right-hand digit*** of the partial dividend 49

e)

$$\begin{array}{r} \cancel{0}\;\cancel{0}\;3 \\ 4\;2\overline{)\;1\;2\;\mathbf{8}} \end{array}$$

42 can't go into 1, nor 12. **Write 0 above 1, & 2.**
128 divided by 42 is 3 r 2. **Write 3 above 8, *the right-hand digit*** of the partial dividend 126.
Again, **drop the two zeros that come before the whole number.**

General Process For Dividing Numbers - Long Division (Read first p.204)

In general, the process of division **follows repeatedly the sequence of steps** described below:

Step 1. ***Divide.*** **Divide the first partial dividend by the divisor.**
Write the 1st digit of the quotient *above the right-hand digit* of the partial dividend, **if the partial dividend has more than one digit.**

Step 2. ***Multiply.*** **Multiply the divisor by the 1st partial quotient (Step 1).**
Write the product under the 1st partial dividend.

Step 3. ***Subtract.*** **Subtract the product (Step 2) from the 1st partial dividend.**
Write the *remainder (or difference)* under the product.

Step 4. ***Bring down.*** **Bring down the next lower number from the dividend.**
The remainder and the number becomes the 2nd partial dividend.

Follow the same steps described above for the 2nd partial dividend.

Note: Division involving **"trial quotient"**, or **"zeros in the quotient"** **does not always follow strictly** the order of steps given above.

Things To Remember In Dividing Larger Numbers

- **Estimate The Final Quotient First** (p.119). Estimate the *approximate* size of the final quotient - will it be **in the ones? in the tens? in the hundreds?...** Give the estimate in round numbers: 10, 300, etc. Compare your estimate with the final answer.

- **The Remainder Must be Smaller Than The Divisor** (p 200). At any point in the process of division, if the remainder is **equal to or larger than** the divisor, **the next larger number** should be tried as a partial quotient.

- **Writing The Remainder In The Final Quotient** (p.200). If a division has a remainder at the end, write the quotient, then **write a small letter "r" to the right, and then write the remainder.** For example: 128 ÷ 3 = 42 r2.

  (Remember that remainder can be expressed as a fraction, a decimal, or even round-off It depends on the nature of the problem. See p.200)

- **Check The Division At The End.** Use multiplication to check division.

  a) Quotient **Without** Remainder:

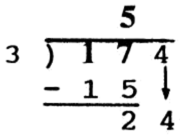
  (Divisor) x (Quotient) = (Dividend)

  b) Quotient **With** Remainder:

  (Divisor) x (Quotient) + (Remainder) = (Dividend)

One-Digit Divisor - Long Division `(Read first "General Process..." p.206)`

(a)



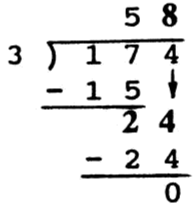
(a) **Steps 1. Divide.** **17 ÷ 3.** (Since 1 can't be divided by 3)
How many 3's are in 17? 5 with r2
Write 5 above 7, the right hand digit of the partial dividend 17.

2. Multiply. **3 x 5 = 15**
(The divisor) x (1st partial quotient)
Write 15 under 17, the partial dividend.

3. Subtract. **17 - 15 = 2.**
(1st partial dividend) - (The product)
Write 2, the remainder, **under 5.**

4. Bring down. Bring down 4 from the dividend.
Write 4 next to 2, the remainder.
24 becomes the 2nd partial dividend.

(b)



(b) **Steps 1. Divide.** **24 ÷ 3.**
Ask: How many 3's are in 24? 8
Write 8 above 4, the right hand digit of the partial dividend.

2. Multiply. **3 x 8 = 24.**
Write 24 under 24.

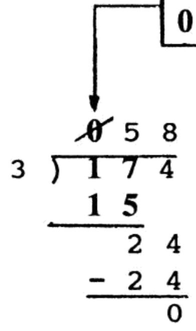
3. Subtract. **24 - 24 = 0.**

Let's go over the last division:

Estimate first (Read "How-To Estimate" p.119).

* Divide: 174 - 3. Think: 170 ÷ 3.
* We know: 3 x 50 = 150; and 3 x 60 = 180.
* 170 is between 150 and 180, **the quotient should be between 50 and 60.**

When 1 can't be divided by 3, we are supposed to write 0 above 1, but we omit it because 058 = 58 (p.66)



17 divided by 3 is 5 with remainder 2. **It is actually 170 ÷ 3 equals 50 with remainder 20.**

3 x 5 = 15. **It is actually 3 x 50 = 150 with 0 in ones place omitted.**

Check: Always check on the work at the end.
3 x 58 = 174 (divisor x quotient = dividend)
Division should begin with estimating and end with checking.

Remember: **Zeros before whole numbers are omitted.** ***But zero(s) as place holder(s) in the quotients can not, and must not, be omitted*** (p.107)

One-Digit Divisor - Short Division or Invert Short Division

With a little practice, you can do division with one-digit divisor (even simple two-digit divisor) **mentally.** The method is called **short division** (a) or **invert short division** (b). For example, let's divide 8706 ÷ 6:

(a)

$$\begin{array}{r} 1\;\;4\;\;5\;\;1 \\ 6\;\overline{)\;8_{2}\;7_{3}\;0\;\;6} \end{array}$$

Begin at the left the thousands place.

a. **6 divided by 8 = 1. Write 1 above (or below) 8,** carry the remainder 2. (8 - 6 = 2)

Before you get used to the method, you may want to jot down the remainder, as shown, to remind you.

or

(b)

$$\begin{array}{r} 6\;\underline{)\;8\;\;7\;\;0\;\;6} \\ 1\;\;4\;\;5\;\;1 \end{array}$$

b. **27 divided by 6 = 4. Write 4 above (or below) 7,** carry the remainder 3. (27 - 24 = 3)

c. **30 divided by 6 =** 5. **Write 5 above (or below) 0,** carry the remainder 0. (30 - 30 = 0)

d. **6 divided by 6 = 1. Write 1 above (or below) 6.**

Example (b) is called invert short division because **the division symbol is written upside down with the quotient under the dividend.**

Connection: When studying *fractions*, you need this skill **in prime factorization. (p.297)**

Division With Numbers Ending In Zeros **(Review first "Dividing By 10..." p.179)**

a)

$$\begin{array}{r} 4 \\ 20\overline{)\,80} \\ -\,80 \\ \hline 0 \end{array}$$

- **Divide.** 8 divided by 2 is 4. 80 divided by 20 is 4. Write the quotient 4 above 0.
- **Multiply.** 20 x 4 = 80. Write 80 under 80.
- **Subtract.** 80 - 80 = 0.

$$\begin{array}{r} 4 \\ 2\cancel{0}\overline{)\,8\cancel{0}} \end{array}$$

Shortcut: Cancel the zero *in both* the divisor and dividend. Then divide.

b)

$$\begin{array}{r} 60 \\ 40\overline{)\,2400} \\ -\,240 \\ \hline 00 \end{array}$$

- **Divide.** 24 divided by 4 is 6. 240 divided by 40 is 6. Write the first digit of the quotient in tens place above 0.
- **Multiply.** 40 x 6 = 240.
- **Subtract.** 240 - 240 = 0. Add 0 to 6.

$$\begin{array}{r} 60 \\ 4\cancel{0}\overline{)\,240\cancel{0}} \end{array}$$

Shortcut: Cancel the zero *in both* the divisor and dividend. Then divide.

Rule: You can cancel *the same number of zeros in both* the divisor and dividend.
You can cancel only the zero(s) that are at the end of the numbers.

Zeros In The Quotient (Read first "General Procedure..." p.206)

Study carefully the following two examples and **notice how zeros in the quotient occur:**

$$\begin{array}{rrrrr} & & 2 & \mathbf{0} & \\ 4 &) & \overline{8} & \overline{2} & \overline{4} \\ & - & \underline{8} & \downarrow & \\ & & & 2 & \end{array}$$

Estimate:

* Think 4 x ( ) = 800. 4 x 200 = 800.
The quotient should be about 200.

Steps 1. 8 divided by 4 = 2. Write 2 above 8.
2. Multiply 4 by 2 = 8. Write 8 under 8.
3. Subtract 8 from 8 = 0. Write nothing.
4. ***Bring down*** 2 from the dividend.
2 can not be divided by 4, *write 0 above 2.*
0 is a place holder

$$\begin{array}{rrrrr} & & 2 & \mathbf{0} & 6 \\ 4 &) & \overline{8} & \overline{2} & \overline{4} \\ & - & \underline{8} & \downarrow & \downarrow \\ & & & 2 & 4 \\ & & - & \underline{2} & \underline{4} \\ & & & & 0 \end{array}$$

4. ***Bring down*** 4 from the dividend.
24 becomes the 2nd partial dividend.

Steps 1. 24 divided by 4 = 6. Write 6 above 4.
2. Multiply 4 by 6 = 24. Write 24 under 24.
3. Subtract 24 - 24 = 0.

Note: If it were 82 - 4, the quotient would have been 20 with r2.

Zeros In The Quotient

$$\begin{array}{rrrrr}
 & & 3 & \mathbf{0} & 9 \\ \cline{2-5}
22\,\big) & 6 & 7 & 9 & 8 \\
 & -6 & 6 & \downarrow & \downarrow \\ \cline{2-3}
 & & 1 & 9 & 8 \\
 & & -1 & 9 & 8 \\ \cline{3-5}
 & & & & 0
\end{array}$$

Estimate:

* Think: 6000 ÷ 20 or 600 ÷ 2 = 300.

* The quotient should be about 300.

Steps

1. 22 into 67 = 3r1 Write 3 above 7.

2. 22 x 3 = 66. Write 66 under 67

3. 67 - 66 = 1 Write 1 under 6

4. ***Bring down 9.*** Write 9 next to 1.

19 < 22 ***Write 0 above 9***

4. ***Bring down 8.*** Write 8 next to 9.

198 is the 2nd partial dividend.

Check:

$$\begin{array}{rrrr}
 & 3 & 0 & 9 \\
\times & & 2 & 2 \\ \hline
 & 6 & 1 & 8 \\
+6 & 1 & 8 & \downarrow \\ \hline
6 & 7 & 9 & 8
\end{array}$$

Steps

1. 22 into 198 = 9. Write 9 above 8.

2. 22 x 9 = 198. Write 198 under 198.

3. 198 - 198 = 0. Write 0 under 8.

Zeros In The Quotient occur when the partial dividend is less than the divisor, and you have to bring down numbers from the dividend for **two consecutive times** - one after another.

Two-Digit Divisor - Trial Quotients (Review first "General Process..." p.206)

When divided by two-digit or larger divisors, you may have to start out with **"trial quotients"** by estimating. Example, divide 1102 - 38:

Trial quotient ──┐
```
             ↓
             3
  3 8 ) 1 1 0 2
      - 1 1 4
```

Trial: * Divide 110 - 38.
Think 11 divided by 3 = 3 r2. Try 3.
* Multiply 38 x 3 = 114.
* **Compare 114 with 110: 114 > 110.**
3 is too big. **Try 2.**

```
            2 9
  3 8 ) 1 1 0 2
        - 7 6 ↓
          3 4 2
        - 3 4 2
              0
```

Steps **1.** Write 2 above 0.
2. Multiply 38 x 2 = 76.
3. Write 76 under 10 and subtract.
4. Bring down 2.

Steps **1.** Divide 342 - 38. **Think 36 ÷ 4 = 9**
Write 9 above 2.
2. Multiply 38 x 9 = 342.
3. Write 342 under 342 and subtract.

Note: You know the trial quotient (estimate) is too large, if you can't subtract; and too small when the remainder (difference) is larger than the divisor

Trial Quotient: If we have noticed that **38 is close to 40,** we would have used 40 as a trial divisor. 110 ÷ 40 would be **2 instead of 3.** ***To keep your work neat and clean, write on a scratch paper while computing trial quotient.***

```
3 8 ) 1 1 0 2
```

110 can be divided by 38, but not 11. A partial dividend must be **equal to or larger than** the divisor. **(Please read p.122)**

```
            2
3 8 ) 1 1 0 2
```

The first digit of the quotient (2) must be **above the right-hand digit** of its partial dividend (110).

```
   tens ────┐
            2
3 8 ) 1 1 0 2
      - 7 6
```

76 = 38 x 2. **Actually 38 x 20 = 760 with 0 in the ones place omitted.** To place 76 under 11 instead of 10 would make 76 equals to 7600, 10 times larger.

Remember: Divisions with "Trial Quotients" require ***the skill of estimating*** and ***some guessing.*** With practice, you will be able to recognize the correct partial quotient without having to go through several trial quotients.

Three-Digit Divisor - Trial Quotient (Review first pp.206-207)

Estimate: Think: 90,000 ÷ 300 = 300. The quotient should be about 300.

```
                2
        _________
2 7 3 ) 8 5 1 7 6
      - 5 4 8
      _______
        3 0 3
```

Trial: * 851 divided by 273. **Think: 8 divided by 3 = 2 r2**
* **Try 2.** 274 x 2 = 548.
* 851 - 548 = 303. **Compare: 303 > 273.**
2 is too small, **try 3.**

```
              3 1 2
        ___________
2 7 3 ) 8 5 1 7 6
      - 8 1 9 ↓ |
      _______   |
          3 2 7 |
        - 2 7 3 ↓
        _______
            5 4 6
            5 4 6
          _______
                0
```

Steps 1. 273 into 851 = 3 r32. Write 3 above 1.
2. 273 x 3 = 819. Write 819 under 851.
3. 851 - 819 = 32. Write 32 under 19.
4. Bring down 7. Write 7 next to 32.

Steps 1. 273 into 327 = 1 r54 Write 1 above 7.
2. 273 x 1 = 273. Write 273 under 327.
3. 327 - 273 = 54. Write 54 under 73.
4. Bring down 6. Write 6 next to 54.

Steps 1. 273 into 546 = 2. Write 2 above 6.
2. 273 x 2 = 546. Write 546 under 546.
3. 546 - 546 = 0.

Finding Averages

Problem: On a recent family vacation, the Smiths kept the record of the distance the family travelled each day as follows:

First day - 300 miles
Second day - 250 miles
Third day - 300 miles
Fourth day - 350 miles
Fifth day - 200 miles

And you want to know the **average distance** the Smiths family travelled a day during the entire trip.

Solution: **Step 1. Adding together the numbers of miles covered in 5 days:**

① ② ③ ④ ⑤ ← Numbers of Addend
300 + 250 + 300 + 350 + 200 = 1,400 miles ← Sum

Step 2. Divide the sum by the number of addends: (here the total number of days).

1,400 ÷ 5 = 280 miles ← **Average**

Answer: The Smiths family travelled **on an average of 280 miles a day.**

To Find Averages: Add the numbers together and then divide the sum by the number of addends.

Order Of Operations (Combined Operations)

We know how to work with addition, subtraction, multiplication, and division separately. But what should we do when **more than one operations** are used in one number sentence? For example:

(a) Is $20 + 5 \times 4 = 100$ or 40?
(b) Is $(20 + 5) \times 4$ the same as $20 + 5 \times 4$?
(c) Is $40 \div 8 \times 5 = 25$ or 1?
(d) $[12 - (2 \times 3)] - 6 = ?$
(e) $4 + 9 \div 3 \times 2^2 = ?$

To prevent different answers to the same arithmetic problems, the general rule of procedure has been internationally accepted in working with combined operations. The rule of procedure is given below:

Order Of Operations:

1st. Work inside the grouping symbols. When there are more than one grouping symbols as in example (d), work from the inside out.
2nd. Work on the exponents as in example (e).
3rd. Do all the multiplication and division, from left to right, in the order in which they appear.
4th. Do all the addition and subtraction, from left to right, in the order in which they appear.

Examples Of Order Of Operations

Let's apply the rule to the examples given on the previous page. We assign numbers ①②③ above the operation symbols to indicate the order of operations:

(a) $20 \overset{\textcircled{2}}{+} 5 \overset{\textcircled{1}}{\times} 4 \longrightarrow 20 + 20 = 40$ (not 100)

(b) $(20 \overset{\textcircled{1}}{+} 5) \overset{\textcircled{2}}{\times} 4 \longrightarrow 25 \times 4 = 100$ (not 40)

(c) $40 \overset{\textcircled{1}}{\div} 8 \overset{\textcircled{2}}{\times} 5 \longrightarrow 5 \times 5 = 25$ (not 1)

(d) $[12 \overset{\textcircled{2}}{-} (2 \overset{\textcircled{1}}{\times} 3)] \overset{\textcircled{3}}{-} 6 \longrightarrow [12 \overset{\textcircled{2}}{-} 6] \overset{\textcircled{3}}{\div} 6 \longrightarrow 6 - 6 = 1$

(e) $4 \overset{\textcircled{4}}{+} 9 \overset{\textcircled{2}}{-} 3 \overset{\textcircled{3}}{\times} \overset{\textcircled{1}}{2^2} \longrightarrow 4 \overset{\textcircled{4}}{+} 9 \overset{\textcircled{2}}{-} 3 \overset{\textcircled{3}}{\times} 4 \longrightarrow 4 \overset{\textcircled{4}}{+} 3 \overset{\textcircled{3}}{\times} 4 \longrightarrow 4 + 12 = 16$

(f) $48 \overset{\textcircled{1}}{\div} 6 \overset{\textcircled{2}}{\times} 3 \overset{\textcircled{4}}{-} 4 \overset{\textcircled{3}}{\times} 6 \overset{\textcircled{5}}{+} 5$ Can you try this one? The answer is 5.

Summary
(Division)

- Division is a repeated subtraction and an inverse operation of multiplication. The division properties are also related to that of multiplication.
- The division facts are the reverse of multiplication facts. The quotient and the divisor are the factors, and the dividend is the product. Because of this, we can use the multiplication table to find the quotient, a missing factor
- There are three ways of writing division. Fraction form is one of them. Since division by zero is impossible, fractions can not have "0" as the denominator
- The remainders in quotients can be written as a whole number, a decimal, or a fraction. However, in real life, we either drop the remainder or increase the quotient by 1 depending on the situation.
- Division, unlike the other three operations, begins at the left. In general, division follows repeatedly 4 steps divide, multiply, subtract, and bring down. If the partial dividend has more than one digit, the partial quotients are always placed right above the right-hand digit of the partial dividend.
- When more than one operations are present in a number sentence, follow the rule on "order of operations" in your computation.

the family **MATH.** companion

ARITHMETIC—THE FOUNDATION OF MATH

Ruth C. Sun

Table of Contents

Decimal Point & Decimal Numbers (Review "Place Value Of Decimals" p.50)

Decimal Point
↓

The Decimal point ***separates*** the whole number part from the fractional part (See next page.)

a)
.25
.250
.2500

Decimals are numbers that are ***less than 1.*** They have digits **only to the right** of decimal point. These digits are called ***decimal digits.***

Note: .25 = .250 = .2500 = .25000. (p.67)

b)
12
12.0
12.00
12.000

Whole Numbers are numbers that ***equal to*** or ***greater than 1.*** They have digits **only to the left** of the decimal point. In whole numbers the decimal point is usually omitted. (See p.228)

Note: 12 = 12.0 = 12.00 = 12.000.

c)
6.85
6.850

Mixed Decimals are numbers that have digits on **both sides** of the decimal point. They are **made up** of:

(Whole Number) + (Decimal)

Remember: Zeros may be added (or dropped) ***at the end*** of a decimal ***without*** **changing its value.**

Decimal Digits Or Fractional Part (See also "Decimals & Decimal Fractions" p.98)

Keep in mind that **decimals** and **fractions** are ***two different ways*** **of writing the same number.** For example, we can call ".526" the decimal digits or the fractional part of the number 13.526 as seen below:

↓
.526 ← **Decimal digits** or **Fractional part** of 13.526
↓
13.526 ← **A mixed decimal** (= **whole number + decimal**)

$13\frac{526}{1000}$ ← **A mixed numbers** (= **whole number + fraction**)

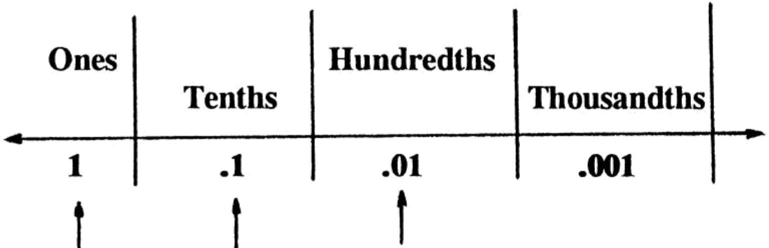
Decimal Digits vs. Digits (See also p.56)

- **Decimal digits** - The words are used with ***decimal numbers.***
  Example: We say "**12.*4518***" has **4 decimal digits.**"

- **Digits** - The word is used with ***whole numbers.***
  Example: We say "**2743** is a **4-digit number.**"

Note: It is important that you learn to read and write decimals correctly. Read page 58.

Borrowing & Carrying With Decimal Numbers



Read first **"Decimal System"**
"Carrying" & **"Borrowing"**
(See pp.140, 158)

One hundredth (.01) is **10 times larger** than **one thousandth** (.001), the digit to its right.

- * ***Borrowing*** **1** hundredth, you get **10** one thousandths.
- * ***Carrying*** **10** one thousandths over becomes **1** hundredth.

One tenth (.1) is **10 time larger** than **one hundredth** (.01), the digit to its right.

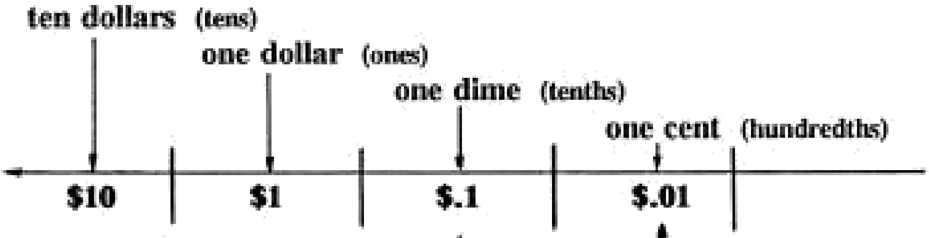
- * ***Borrowing*** **1** tenth, you get **10** one hundredths (.01).
- * ***Carrying*** **10** one hundredths over becomes **1** ten.

One (1) is **10 times larger** than **one tenth** (.01), the digit to its right.

- * ***Borrowing*** **1**, you get **10** one-tenths (.1).
- * ***Carrying*** **10** one tenths over becomes **1**.

Decimal System & U.S. Money System (See also "Decimal & Money" p.59)

Our money system *works exactly* like decimal system, except it uses only two decimal places - the **tenths (dime)** & the **hundredths (cent).**

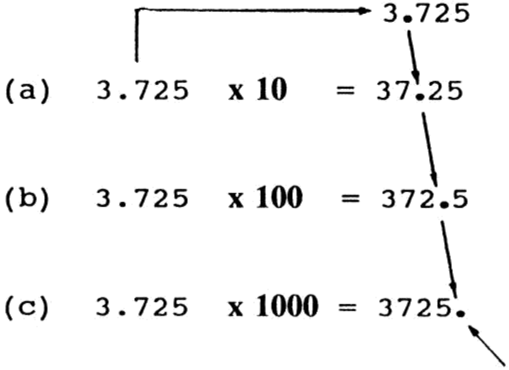


One cent is **one tenth (1/10)** of one dime.
1 dime equals **10 one-cent.**
One cent is **one hundredth** of one dollar.
1 dollar equals **100 one-cent.**

Note: $1 = $1.00; $.1 = $.10

One dime is **10 times** more than one cent.
10 one-cent can be exchanged for 1 dime.
One dime is **one tenth (1/10)** of one dollar.
10 dimes can be exchanged for 1 dollar

Multiplying Decimals by 10, 100, etc. (See also p.68)



(a) 3.725 x 10 = 37.25

(b) 3.725 x 100 = 372.5

(c) 3.725 x 1000 = 3725.

In writing whole numbers, we omit the decimal point.

Think!

1 Do you divide or multiply when you move the decimal point to the right?
2 Does the number become smaller or larger when the decimal point is moved to the right?
3 Are the number of zeros in the multiplier related to the number of places the decimal point is moved?

General Rule:

When multiplying by 10, 100, etc., move the decimal point *to the right* the same number of places as there are zeros in the multiplier.

Example:
* Multiplying by 10 - **1 zero** - move **1 place.**
* Multiplying by 100 - **2 zeros** - move **2 places.**
* Multiplying by 1000 - **3 zeros** - move **3 places.**

Dividing Decimals By 10, 100, etc. (See also p 69)

In whole number, the decimal point is at the right of ones.

8265.

(a) 8265. ÷ 10 = 826.5

(b) 8265. ÷ 100 = 82.65

(c) 8265. ÷ 1000 = 8.265

Think!

1 Do you divide or multiply when you move the decimal point to the left?

2. Does the number become larger or smaller when the decimal point is moved to the left?

3 Are the number of zeros in the divisor related to the number of places the decimal point is moved?

General Rule:

When dividing by 10, 100, etc., move the decimal point *to the left* as many places as there are zeros in the divisor.

Example: * Dividing by 10 - **1 zero** - move **1 place.**
* Dividing by 100 - **2 zeros** - move **2 places.**
* Dividing by 1000 - **3 zeros** - move **3 places.**

Estimating Sums of Decimals & Money (Review "Rounding" p 30)

We estimate the sums/differences of decimals in the same way we estimate the sums/differences of whole numbers (See p.116)

a) Estimate 1.5 + 0.75 + 6.18.

1.5
0.75
+ 6.38
7.00

.5, .75,
& .38 is
about 1.5

7 + 1.5 = 8.5

Method 1. Using Front-end digits & Adjusting
Write the addends vertically with the place values lined up correctly.
1st. Add the whole numbers part: 7.
2nd. Estimate the sum of the decimal part and add to the initial estimate.
8.5 is the adjusted estimate.

b) Estimate $16.58 + $7.89.

Front digits	Rounding
$16.58	$17.00
+ 7.89	+ 8.00
$23.00	$25.00

Method 1. Using Front-end digits
Add the whole numbers without adjustment.
Method 2. Using Rounding
Rounding to the nearest whole number, ones digit, **(same place)** and add.

The actual sum is more than $23.00 and less than $25.00.

Estimating Differences of Decimals & Money (Review first p.117)

Remember there are many ways to estimate. Choose a method that is the easiest for the situation and also gives a closer estimation. Think!

a) Estimate 21.75 - 2.04 (**Clue:** Think of 22 - 2.)

$$\begin{array}{r} 21.75 \\ -\ \ 2.04 \\ \hline \end{array} \qquad \begin{array}{rl} 22.00 & \text{(up)} \\ -\ \ 2.00 & \text{(down)} \\ \cline{1-1} 20.00 & \end{array}$$

Rounding is a better method. Remember that you round the minuend up and subtrahend down, so the actual difference should be less than 20.

b) Estimate \$75.83 - \$25.05 (**Clue:** Both have 5 in the ones digit.)

$$\begin{array}{r} \underline{\textbf{Front Digits}} \\ \\ \$75.83 \\ -\ 25.05 \\ \hline \$50.00 \end{array} \qquad \begin{array}{r} \underline{\textbf{Rounding}} \\ \\ \$80.00 \\ -\ 30.00 \\ \hline \$50.00 \end{array}$$

Since .83 > .05, the actual difference is more than \$50.

Front Digits (left). Subtract the whole number part.
Rounding (right). Round the numbers to the largest place, to the nearest tens, and subtract. Both methods give the same estimate.

> The advantage of using front digits is that we can compare the decimal digits and adjust the estimate.

Estimating Products of Decimals & Money (Review first p.118)

The method is the same as estimating products of whole numbers The key is changing one factor or both to one-digit-number(s) with the remaining digits zeros so that we can multiply mentally.

a) Estimate 28.56 x 2.79

28.56	up	30.00
x 2.79	up	x 3.00
		90.00

Round each factor to **its own** largest place and multiply. Since both factors were rounded up, **adjustment is needed.** The actual product **is less than** 90.

b) Estimate 0.49 x 5.18

0.49	up	0.5
x 5.18	down	x 5
		2.5

Round 0.49 up and round 5.18 down and multiply. No adjustment is needed. It is a reasonable estimate.

c) Estimate $1.49 x 35

(Change both to next whole number.)

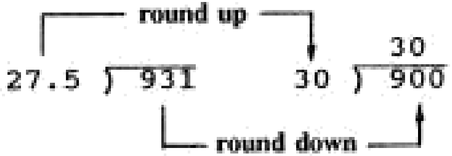
$1.49	up	$2.00
x 35	up	x 40
		$80.00

It is better to overestimate if you are estimating to see whether you have enough money to make a purchase.

Estimating Quotients Of Decimals & Money

In estimating decimal quotients, we use the **multiplication/division facts** to find the compatible numbers to replace the divisor or the dividend, or both, so that we can compute mentally.

a) Estimate 931 - 27.5



* **Dividing Whole Numbers by Decimals:**
(**Clue:** 9 - 3 = 3 and 90 - 30 = 3)
Round up the divisor and round down the dividend. The actual quotient should be larger than the estimate. Do you know why?

b) Estimate $1.98 - 3

$$3\overline{)\$1.98} \longrightarrow \overset{\$.60}{3\overline{)\$1.80}}$$

* **Dividing Decimals by Whole Numbers:**
(**Clue:** 1.8 ÷ 3 or 2.1 ÷ 3)
Change the dividend to a **lower number** $1.80. The actual quotient should be larger than $.60.

$$3\overline{)\$1.98} \longrightarrow \overset{\$.70}{3\overline{)\$2.10}}$$

Change the dividend to a **higher number** $2.10. The actual quotient should be smaller than $.70.

Summary

(Introduction)

* The decimals are numbers that are between 0 and 1 Decimals have digits only to the right of the decimal point while whole numbers have digits only to the left of the decimal point.
* The mixed decimals are made up of whole numbers and decimals. They have digits on both sides of the decimal points.
* The digits to the right of the decimal point are called the decimal digits or the fractional part of a number
* Carrying and borrowing with decimal numbers follows the same rules as that of whole numbers.
* To multiply a decimal by the powers of 10, we move the decimal point to the right the same number of places as there are zeros in the multiplier
* To divide a decimal by the powers of 10, we move the decimal point to the left the same number of places as there are zeros in the divisor
* Make it a habit to estimate the answer before computation. Then check your answer against the original estimation

Part III. Decimal Operations

B. Addition
C. Subtraction
D. Multiplication

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Part III. Decimal Operations - Addition

Part III. Decimal Operations - Subtraction

Part III. Decimal Operations - Multiplication

Adding Decimal Numbers (Review first "Addition of Whole Numbers" p.143)

General Rule: * First write the numbers vertically **with the decimal points lined up** in a column. * **Add zeros** if necessary, so that all addends will have same number of the decimal digits. * Then **add the decimals just as with whole numbers.** * Finally, bring the decimal point **straight down to the sum.**

Example: Add 74.3 + 5.168

$$75 + 5 = 80$$

Step 1. Estimate. The sum is about 80.

$$\downarrow\downarrow$$

$$\begin{array}{r} 74.3 \\ +\ \ 5.168 \\ \hline \end{array}$$

Step 2. Write the addends under each other with the digits of the **same place value** lined up. When you do, the decimal points lined up also.

$$\downarrow\downarrow$$

$$\begin{array}{r} 74.300 \\ +\ \ 5.168 \\ \hline \end{array}$$

Step 3. To avoid error, add two zeros to the first addend after the decimal digit 3, so that both addends will have the **same number of the decimal digits.**

$$\begin{array}{r} 74.300 \\ +\ \ 5.168 \\ \hline 79.468 \end{array}$$

Step 4. Add the decimal numbers just as you would with whole numbers. Then bring the decimal point straight down the column into the sum.

Remember: Adding zeros *after* a decimal number *does not change* its value.

Adding Whole Numbers & Decimal Numbers (Review "Carrying..." p.140)

Example: Add .427 + 7.61 + 45

$$45 \longrightarrow 45.$$

Step 1. Put a decimal point to the right of the whole number - at the right of the ones digit.

$$\begin{array}{rl} & .427 \\ 7 & .61 \\ +\ 45 & . \\ \hline \end{array}$$

Step 2. Write the addends vertically with the digits lined up correctly or with the decimal points lined up under each other.

$$\begin{array}{rl} & .427 \\ 7 & .610 \\ +\ 45 & .000 \\ \hline \end{array}$$

Step 3. To avoid error, add zeros to addends so that each addend has the same number of decimal digits or places - 3 decimal places.

$$\begin{array}{rl} {\scriptstyle 1} & \\ & .427 \\ 7 & \ 610 \\ +\ 45 & .000 \\ \hline 53 & .037 \end{array}$$

Step 4. Add as you would with whole numbers with carrying in two places: from tenths to ones, and again from ones to tens. Then, bring the decimal point straight down to the sum.
Remember: Carrying is always from right to left.

Remember: The **properties, rules,** and **shortcuts,** that apply to the addition of whole numbers, are also **applicable** to the addition of decimal numbers.

Subtracting Decimal Numbers (Review first "Borrowing .. " p.158)

General Rule: * First write the numbers vertically with **the decimal points lined up.** * **Add zeros** if necessary so that each number has the same number of the decimal digits. * Then subtract decimals just **as with whole numbers.** * Finally, bring the decimal point **straight down to the difference.**

Example: Subtract 5.38 - 0.921

$$\begin{array}{r} 5.38\downarrow \\ -\ 0.921 \\ \hline \end{array}$$

Step 1. Write the numbers vertically with the decimal points lined up. **The Larger number always at the top.**

$$\begin{array}{r} 5.380 \\ -\ 0.921 \\ \hline \end{array}$$

Step 2. Add zero if necessary, so that both numbers have the same number of decimal digits.

Step 3. Subtract as if they were whole numbers.

$$\begin{array}{r} 5.3\overset{7}{\cancel{8}}\overset{10}{0} \\ -\ 0.921 \\ \hline 4.459 \end{array}$$

* Subtract the thousandths: 0 - 1 = impossible.
  Borrow 1 hundredth that makes 10 thousandths. 10 - 1 = 9.
* Subtract the hundredths: 7 - 2 = 5.
* Subtract the Tenths: 3 - 9 = impossible.
  Borrow 1 from the ones that makes 10 tenths. 13 - 9 = 4.
* Subtract the ones: 4 - 0 = 4.
* **Bring the decimal point straight down to the difference.**

Note: The difference should have the same number of decimal digits as those of the largest number

Subtracting Money (See also "Decimals & Money" p.59)

Except for the dollar sign, we subtract (or add) money **in exactly the same way** as we subtract (or add) decimal numbers. Decimals can have many decimal digits, but money uses **only two decimal places: the dime and the cent.**

Example: Subtract \$10 - \$3.24.

$\$10 \longrightarrow \$10.00$

Step 1. Put a decimal point to the right of the whole number followed **by two zeros.**

$$\begin{array}{r} \$10.00 \\ \underline{-\ \$3.24} \end{array}$$

Step 2. Write the number vertically with the decimal points lined up.

$$\begin{array}{rccccc} & \scriptstyle 9 & & & \scriptstyle 9 & \scriptstyle 10 \\ \$ & \cancel{1} & 0 & . & 0 & 0 \\ -\ \$ & & 3 & . & 2 & 4 \\ \hline \$ & & 6 & . & 7 & 6 \end{array}$$

Step 3. To subtract, we have to borrow across the zeros - the dime & the dollar places.
Replace the 10 dollars with the following:
9 dollar, 9 dimes, and 10 cents.

Then begin at the right and subtract.
* Subtract the cents (hundredths): 10 - 4 = 6.
* Subtract the dimes (tenths): 9 - 2 = 7.
* Subtract the dollars (ones): 9 - 3 = 6.
The difference is \$6.76.

Check:

$$\begin{array}{r} \$6.76 \\ +\ \$3.24 \\ \hline \$10.00 \end{array}$$

Multiplying A Decimal By A Whole Number

General Rule: First, multiply decimals just as with whole numbers ignoring the decimal points. Then, count the total number of the decimal places in both the multiplicand and the multiplier combined. Finally, point off from the right in the product *the same number of decimal places.*

Example: Multiply 9.37 x 4

$$\begin{array}{r} {}^{1}\ \ {}^{2}\ \\ 9.37 \\ \times\quad 4 \\ \hline 3748 \end{array}$$

Step 1. Multiply 937 x 4 (Multiply as whole numbers)

1st Multiply 4 x 7 = 28.
Write 8 under 4. Carry 2.

2nd. Multiply 4 x 3 = 12. 12 + 2 = 14.
Write 4 in the same column with 3. Carry 1.

3rd. Multiply 4 x 9 = 36. 36 + 1 = 37.
Write 37 to the left of 4.

$$\begin{array}{rl} 9.37 & \text{(2 places)} \\ \times\quad 4 & \\ \hline 37.\underset{\curvearrowleft}{48} & \text{(2 places)} \end{array}$$

Step 2. Put a Decimal Point in the product:

1st. Count the ***total number*** of decimal places in both factors combined: 2 places.

2nd. Point off ***from the right*** 2 places in the product -- 37.48.

Note: In multiplying decimals, you don't think of the decimal points until you come to the product.

Multiplying A Decimal By A Decimal (Review "General Process..." p.182)

$$\begin{array}{rl} .28 & \\ \times\ .34 & \\ \hline 112 & — 28 \times 4 \\ +\ 84 & — 28 \times 3 \\ \hline 952 & \end{array}$$

Step 1. Multiply 28 x 4.

1st. Multiply 4 x 8 = 32.
Write 2 under 4. Carry 3.

2nd. Multiply 4 x 2 = 8. 8 + 3 = 11.
Write 11 on the left of 2.

Step 2. Multiply 28 x 3.

1st Multiply 3 x 8 = 24.
Write 4 **in the same column** with 3. Carry 2.

2nd. Multiply 3 x 2 = 6. 6 + 2 = 8.
Write 8 on the left of 4.

$$\begin{array}{rl} .28 & \text{(2 places)} \\ \times\ .34 & \text{(2 places)} \\ \hline 112 & \\ +\ 84 & \\ \hline \underbrace{.0952} & \text{(4 places)} \end{array}$$

Step 3. Add the partial products: 952

Step 4. Place the Decimal Point in the Product.

1st. Count the **number of decimal places** in both factors combined: **2 + 2 = 4 places.**

2nd. ***Place a zero before 9*** and **a decimal point before the zero** to make **4 places** in the product.

If not enough decimal places in the product, insert zeros between the decimal point and the first decimal digit.

Summary

(Addition, Subtraction, Multiplication)

- Adding zero(s) after a decimal number does not change the value of the number But inserting zero(s) between the decimal point and the first decimal digit does change the value of the decimal - the decimal becomes smaller
- To add or subtract decimal numbers, write the numbers one under the other with the digits lined up according to their place value. If you do, the decimal points will also be lined up.
- To avoid error, add zeros to the empty places so that every number has the same number of decimal digits. Then add or subtract as you would with whole numbers. At the end, bring the decimal point straight down to the sum or the difference.
- To multiply decimals, you don't have to line up the decimal points. Write the decimals vertically and multiply them as if they were whole numbers. Then count the total number of decimal places in both factors combined. Point off from the right the same number of places in the product. If not enough decimal places in the product, add zeros between the decimal point and the first decimal digit.

Part III. Decimal Operations

E. Division

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Dividing Whole Numbers - Mixed Decimals (Review "Division" p.96)

You know a remainder appears when the numbers *can not be divided evenly*. **By adding a decimal point and zero(s) to the dividend, you can continue to divide.** And the quotient will be a mixed decimal. **Example:** Divide 34 ÷ 5.

$$\begin{array}{r} 6 \\ 5\,\overline{)\,34} \\ -\,30 \\ \hline 4 \end{array}$$

Steps **1.** Divide: 34 - 5 = 6 r4 Write 6 above 4.
2. Multiply: 5 x 6 = 30. Write 30 under 34.
3. Subtract: 34 - 30 = 4. **4 is smaller than 5.**

To write the remainder as a decimal:
*** Add a decimal point and a zero to the dividend.**
*** Put a decimal point in the quotient right above the one in the dividend. Then continue to divide.**

$$\begin{array}{r} 6.8 \\ 5\,\overline{)\,34.0} \\ -\,30 \\ \hline 4\;0 \\ -\,4\;0 \\ \hline 0 \end{array}$$

Steps **1.** Bring down 0. **40 is the 2nd partial dividend.**
2. Divide: 40 ÷ 5 = 8. Write 8 above 0.
3. Multiply: 5 x 8 = 40. Write 40 under 40.
4. Subtract: 40 - 40 = 0 Write 0 under 0.

Rule: The number of decimal places in the quotient *should equal* the number of decimal places in the dividend.

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Dividing Whole Numbers - Mixed Decimals **(Review "Division" p.96)**

You know a remainder appears when the numbers *can not be divided evenly.* **By adding a decimal point and zero(s) to the dividend, you can continue to divide.** And the quotient will be a mixed decimal. **Example:** Divide 34 ÷ 5.

```
      6
5 ) 34
   - 30
   ----
      4
```

Steps 1. Divide: 34 - 5 = 6 r4 Write 6 above 4.
2. Multiply: 5 x 6 = 30. Write 30 under 34.
3. Subtract: 34 - 30 = 4. **4 is smaller than 5.**

To write the remainder as a decimal:
*** Add a decimal point and a zero to the dividend.**
*** Put a decimal point in the quotient right above the one in the dividend. Then continue to divide.**

```
      6.8
5 ) 34.0
   - 30
   ----
      4 0
    - 4 0
    -----
        0
```

Steps 1. Bring down 0. **40 is the 2nd partial dividend.**
2. Divide: 40 ÷ 5 = 8. Write 8 above 0.
3. Multiply: 5 x 8 = 40. Write 40 under 40.
4. Subtract: 40 - 40 = 0 Write 0 under 0.

Rule: The number of decimal places in the quotient *should equal* the number of decimal places in the dividend.

Dividing Smaller Numbers By Larger Numbers - Terminating Decimals

Remember, when the dividend is *smaller than* the divisor, like the proper fractions, the quotient will be **a decimal, less than 1.**
Example: Divide 1 ÷ 4. (To change 1/4 to a decimal, we divide 1 - 4.)

```
     0.
 4 ) 1.0
```

1st. 1 ÷ 4 = impossible. **Write 0.** in quotient above 1.
Add a decimal point and a zero to the dividend.

```
     0.2
 4 ) 1.0
   - 8
     2
```

2nd. Divide: 10 ÷ 4 = 2 r2 Write 2 in quotient above 0.
(Treat the new dividend, 1.0, as whole number 10.)
Multiply: 4 x 2 = 8. Write 8 under 0.
Subtract: 10 - 8 = 2. **2 is smaller than 4.**

```
     0.25
 4 ) 1.00
   - 8
     20
   - 20
      0
```

3rd. Add another zero to the dividend.
Bring the zero down. **20 is the 2nd partial dividend.**
Divide 20 ÷ 4 = 5. Write 5 in the quotient.
Multiply 4 x 5 = 20. Write 20 under 20.
Subtract: 20 - 20 = 0.

Rule: Whenever the dividend, remainder included, *is smaller than* the divisor, **add zero(s)** to the dividend or remainder and continue to divide. **Repeat the process until it reaches a "0" remainder** for terminating decimals.

Dividing Smaller Numbers By Larger Numbers - Repeating Decimals

Repeating decimals may occur whether the dividend *is larger*, or *is smaller than* the divisor. Example: Divide 1 - 3. (The fraction 1/3 means 1 - 3.)

$$\begin{array}{r} 0. \\ 3\,\overline{)\,1.0} \end{array}$$

Steps 1. 1 – 3 = impossible. Write 0. above 1.
2. Add a decimal point and zero to the dividend.

$$\begin{array}{r} 0.3 \\ 3\,\overline{)\,1.0} \\ \underline{-\ 9} \\ 1 \end{array}$$

Steps 1. Divide: 10 - 3 = 3 r1 Write 3 above 0.
2. Multiply: 3 x 3 = 9. Write 9 under 0.
3. Subtract: 10 - 9 = 1. Write 1 under 9.

$$\begin{array}{r} 0.33 \\ 3\,\overline{)\,1.00} \\ \underline{-\ 9} \\ 10 \\ \underline{-\ 9} \\ 1 \end{array}$$

Steps 1. Add another 0 to the dividend.
2. Bring down zero. **10 is the 2nd partial dividend.**
3. Divide: 10 ÷ 3 = 3 r1 Write 3 above 0.
4. Multiply: 3 x 3 = 9. Write 9 under 0.
5. Subtract: 10 - 9 = 1. Write 1 under 9.

You can predict that **3 in the quotient will repeat**, because no matter how far we carry out the division, there will be **a remainder 1.** In such case, we simply **write a bar above the digit**, or the block of the digits, **that repeat**. See the example on the left.

$1 \div 3 = 0.3\overline{3}$

$1 \div 3 = 0.33...$

Connection Among Whole Numbers, Mixed Decimals, And Mixed Numbers

In studying math, it is important to see how the different parts are connected. For example, whole numbers, mixed decimals (decimals), and mixed numbers (fractions) are studied under different subjects, but they may all appear in the **division of whole numbers as seen below.** When you divide whole numbers, you will notice that the quotient could be·

(a) **A Whole Number** - If the dividend *is larger than* the divisor and if it *divides evenly* with 0 as the remainder.
Example: 27 - 3 = 9.

(b) **A Mixed Decimal** (A decimal) - If the dividend *is larger than* the divisor and if it *doesn't divide evenly* The quotient may be **a terminating decimal.**
Example: 17 ÷ 5 = 3.4

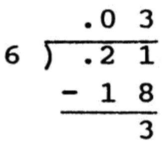
(c) **A Mixed Number** (A fraction) - If the dividend *is larger than* the divisor and if it *doesn't divide evenly.* The quotient may be **a repeating decimal.**
Example: 34 ÷ 7 = 4 6/7

The best way to present a repeating decimal is to write it as a mixed number.

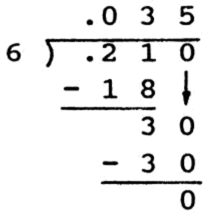
Dividing Decimals By Whole Numbers

The following two examples show you *where* **to place the decimal points** when the dividends are decimals and the divisors are whole numbers.
Example: Divide .21 ÷ 6.

Since 2 can not be divided by 6, write 0 above 2. Remember that 0s after the the decimal point and before the decimal digit are "place holders." They can not be omitted! You see .3 is 10 times larger than .03.



Steps 1. Put a decimal point in the quotient above the decimal point in the dividend, **and followed by 0.**
2. Divide: 21 - 6 = 3 r3 Write 3 above 1.
3. Multiply: 6 x 3 = 18.
4. Subtract: 21 - 18 = 3. 3 is smaller than 6.



Steps 1. Add a zero to the dividend.
2. Bring down 0. **30 is the 2nd partial dividend.**
3. Divide: 30 - 6 = 5. Write 5 above 0.
4. Multiply: 6 x 5 = 30. Write 30 under 30.
5. Subtract: 30 - 30 = 0.

Example: Divide 35.07 - 7 = 5.01

To the left of the decimal point are whole numbers. And 0s before whole numbers are dropped since they have no value: 05 = 5 (See p.66)

```
      05.
7 ) 35.07
  - 35
     0
```

Steps 1. Divide: 3 - 7 = impossible.

2. Divide: 35 ÷ 7 = 5. Write 0 above 3.

3. Multiply: 7 x 5 = 35. Write 5 above 5.

4. Subtract: 35 - 35 = 0. Write 35 under 35. Write 0 under 5.

```
       5.01
7 ) 35.07
  - 35 ↓↓
     0 07
      - 7
        0
```

Steps 1. Put a decimal point after 5.

2. Bring down 0. **Write 0 in the quotient above 0.** (Since 0 ÷ 7 = 0)

3. Bring down 7. **7 is the 2nd partial dividend.**

4. Divide: 7 - 7 = 1. Write 1 above 7.

5. Multiply: 7 x 1 = 7. Write 7 under 7.

6. Subtract: 7 - 7 = 0.

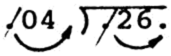
Note: A mixed decimal (35.07) **= whole number** (35) **+ decimal** (.07)

35.07 - 7 = (35 - 7) + (.07 ÷ 7) = 5 + .01 = 5.01

whole number **decimal** **quotient**

Changing Decimal Divisors To Whole Numbers (Review first p.228)

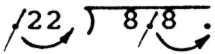
When the divisor is a decimal, **you must make it a whole number** before you can carry out the division. The following shows the how-to:



4) 26

Move the decimal point 2 places to the right:

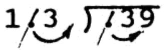
* The divisor: .04 — (x 100) ⟶ 4 (Drop 0)
* The dividend: .26 — (x 100) ⟶ 26



22) 880

Move the decimal point 2 places to the right:

* The divisor: .22 — (x 100) ⟶ 22
* The dividend: 8.8 — (x 100) ⟶ 880 (Add 0)



13) 3.9

Move the decimal point 1 place to the right:

* The divisor: 1.3 — (x 10) ⟶ 13
* The dividend: .39 — (x 10) ⟶ 3.9

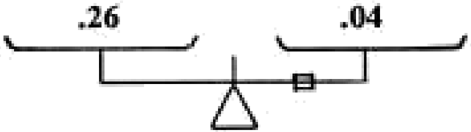
6).018

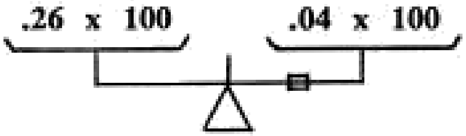
Move the decimal point 1 place to the right:

* The divisor: .6 — (x 10) ⟶ 6
* The dividend: .0018 — (x 10) ⟶ .018

Remember: To move the decimal point to the right is the same as multiplying the number by the powers of 10: 10, 100, etc.

Error To Avoid: In changing decimal divisors to whole numbers, students tend to change the divisor without changing the corresponding dividend. Keep in mind that the relation between the divisor and the dividend is like a balance.





In order to maintain the balance, you must do to the dividend what you have done to the divisor.

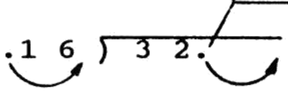
Another way to avoid the error is to **write the division in fraction form:**

$$\frac{\text{dividend}}{\text{divisor}} \qquad \frac{.26}{.04} = \frac{.26 \times 100}{.04 \times 100} = \frac{26}{4}$$

If you multiply the divisor by 100, you must also multiply the dividend by 100. 26/4 is an equivalent fraction of .26/.04. (See p.301)

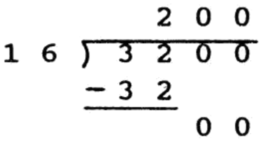
Dividing Whole Numbers By Decimals

Read first the previous two pages before you study the following example. **Example:** Divide 32 - .16



The decimal point of whole numbers is at the right of ones digit.

1st. Make *the divisor* a whole number **by moving the decimal point to the right of 6** - 2 places.



2nd. Move the decimal point of *the dividend* **an equal number of places** - 2 places. **And fill the vacant places with zeros.** Then divide.

Again, we use *fractions* to show the steps given above:

$$\frac{32}{.16} = \frac{32 \times 100}{.16 \times 100} = \frac{3200}{16}$$

32/.16 and 3200/16 **are equivalent fractions** with *different numbers* only.

When we move the decimal point of the divisor *2 places to the right,* we **multiplied the bottom number by 100.** In order to **keep the value** of the original fraction, **we must also multiply the top number (the dividend) by 100.**

Example: Divide 752 - .8 (The division in fraction is 752/.8)

$\underset{\curvearrowright}{.8}\,\overline{)\,752\underset{\curvearrowright}{.\;\;}} = 8\,\overline{)\,7520}$

First. Make the divisor a whole number by moving the decimal points of ***both* the divisor and dividend 1 place to the right.** Then divide.

$$\begin{array}{r} 9\;4\;0 \\ 8\,\overline{)\,7\;5\;2\;0} \\ \underline{-\,7\;2}\downarrow\;\downarrow \\ 3\;2 \\ \underline{-\,3\;2} \\ 0\;0 \\ \underline{-\,0\;0} \\ 0 \end{array}$$

Steps **1.** Divide 75 ÷ 8 = 9 r3 Write 9 above 5.
2. Multiply 8 x 9 = 72. Write 72 under 75.
3. Subtract 75 - 72 = 3. Write 3 under 2.
4. Bring down 2. Write next to 3.

Steps **1.** Divide 32 - 8 = 4. Write 4 above 2.
2. Multiply 8 x 4 = 32. Write 32 under 32.
3. Subtract 32 - 32 = 0. Write 0 under 2.
4. Since 0 ÷ 8 = 0. Write 0 above 0.

Check: Is 752 - .8 = 940? We know **"quotient x divisor = dividend".**

$$\begin{array}{r} \text{Quotient} \\ \underline{\text{x}\;\;\text{Divisor}} \\ \text{Dividend} \end{array} \qquad \begin{array}{r} 940 \\ \underline{\text{x}\;\;\;.8} \\ 752.0 \end{array}\;\begin{array}{l} \\ \text{(1 decimal place)} \\ \text{(1 decimal place)} \end{array} \qquad 752.0 = 752$$

Note: 752 ÷ .8 is 940 and 7520 ÷ 8 is also 940.

Dividing Decimals By Decimals (Review first p.228)

Example: Divide .0021 ÷ .6

```
.6 ).0021  =  6 ).021
```

Move the decimal points of the divisor and the dividend 1 place to the right.

0s to the right of the decimal point, and before the decimal digits are "place-holders." They can not be omitted.

```
     .003
  6 ).021
    - 18
    ----
       3
```

Steps 1. **Put a decimal point in the quotient above the decimal point in the dividend.**

2. Divide 0 ÷ 6 = 0. Write 0 above 0.

3. Divide 2 ÷ 6 = impossible. Write 0 above 2.

4. Divide 21 - 6 = 3 r3. Write 3 above 1.

5. Multiply 6 x 3 = 18. Write 18 under 21.

6. Subtract 21 - 18 = 3. Write 3 under 8.

```
     .0035
  6 ).0210
    - 18
    ----
       30
     - 30
     ----
        0
```

Steps 1. Add 0 to the dividend.

2. Bring down 0. Write 0 next to 3.

3. Divide 30 - 6 = 5. Write 5 above 0.

4. Multiply 6 x 5 = 30. Write 30 under 30.

5. Subtract 30 - 30 = 0. Write 0 under 0.

Example: Divide 49.247 ÷ 1.21

$1.21\overline{)\,49.247} = 121\overline{)\,4924.7}$ Move two places to the right.

Since 4 or 49 can not be divided by 121, we are supposed to put 0s above those two digits. But we drop them because 0s before whole numbers have no value.

$$\begin{array}{r} \cancel{0}\cancel{0}4 \\ 121\overline{)\,4924.7} \\ \underline{-\;484} \\ 84 \end{array}$$

Steps
1. Divide 492 ÷ 121 = 4 r8. Write 4 above 2.
2. Multiply 121 x 4 = 484. Write 484 under 492.
3. Subtract 492 - 484 = 8. Write 8 under 4.
4. Bring down 4. Write 4 next to 8.

$$\begin{array}{r} 40.7 \\ 121\overline{)\,4924.7} \\ \underline{-\;484} \\ 84\;7 \\ \underline{84\;7} \\ 0 \end{array}$$

Steps
1. 84 can't be divided by 121. Write 0 above 4.
2. **Put a decimal point in the quotient right above the decimal point in the dividend.** Bring down 7.
3. Divide 847 ÷ 121 = 7. Write 7 above 7.
4. Multiply 121 x 7 = 847.
5. Subtract 847 - 847 = 0.

Remember: The decimal places of the quotient **must equal** the decimal places of the dividend.

Dividing Money

Dividing money is done ***exactly the same*** as dividing with decimals or whole numbers. Example: Divide $18.00 ÷ 24

Since 0s means no dollars, we drop them but keep the dollar sign and the cent point (decimal point).

$$\begin{array}{r} \$\cancel{0}\cancel{0}.7 \\ 24\,\overline{)\;\$18.00} \\ -16\;8 \\ \hline 1\;20 \end{array}$$

Steps 1. 1 or 18 can't be divided by 24, **write 0s above 1 and 8, and followed by a decimal (or cent) point.**

2. Divide 180 ÷ 24 = 7 r12. **Write 7 in dime place.**
Think: 25 x 4 = 100 & 25 x 8 = 200.

3. Multiply 24 x 7 = 168. Write 168 under 180.

4. Subtract 180 - 168 = 12. Write 12 under 68.

5. Bring down 0. Write next to 2.

$$\begin{array}{r} \$\quad.75 \\ 24\,\overline{)\;\$18.00} \\ -\;16\;8 \\ \hline 1\;20 \\ -\;1\;20 \\ \hline 0 \end{array}$$

Steps 1. Divide 120 ÷ 24 = 5. **Write 5 in cent place.**

2. Multiply 24 x 5 = 120 Write 120 under 120.

3. Subtract 120 - 120 = 0. Write 0 under 0.

The answer is **$.75.**

Note: * The **cent (decimal) point** is located **two places from the right.**
* Whole dollars can be written as either $18 or $18.00.

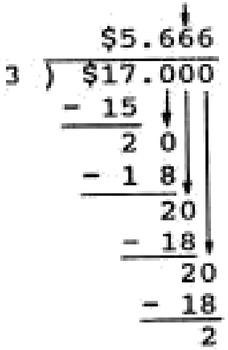
Dividing Money With Repeating Decimals

When dividing with decimals, some quotients appear to continue without end - they are repeating decimals. We use rounding to handle such cases in the following way:

First, Determine an appropriate **number of decimal places** for the quotient.
Then, Divide to **one additional place** - carry the division one extra place.
Finally, **Round** the quotient to the desired (or required) place.

In general, money is rounded to the nearest cent - 2 decimal places.

Example: Divide to the nearest cent: $17 ÷ 3.



1st. Divide. Keep the dollar sign ($) to serve as a reminder.

2nd. **Mark the hundredths (cent) place** and carry the division *one extra place* - to the thousandths place.

3rd. **Round** the quotient **to the nearest hundredth** (cent) place - 2nd place to the right of the decimal point. Since 6 in the thousandths place is greater than 5, **add "1"** to the "cent" place. The answer: $5.67

Summary
(Division)

* In dividing whole numbers, if the divisor is smaller than the dividend, the quotient can be a whole number or a mixed decimals. But if the divisor is larger than the dividend, the quotient will be a decimal - either a terminating decimal or a repeating decimal.

* In division, if the divisor is a decimal, make it a whole number by moving the decimal point to the right of the last digit. Then move the decimal point of the dividend an equal number of places. If the dividend is a whole number, fill the place(s) with zero(s)

* Division with decimal numbers is similar to division with whole numbers. Special attention is directed to the place of decimal points in quotients.

* The decimal point in the quotient should be right above the decimal point in the dividend. And the number of decimal places in the quotient should equal the number of decimal places in the dividend.

* Mixed numbers is the best way to write a repeating decimal. In general, money is rounded to the nearest cents (hundredth place)

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ARITHMETIC—THE FOUNDATION OF MATH

Ruth C. Sun

Part IV. Fraction Operations

A. Introduction

Table of Contents

Common Fractions - Terminating Decimals (See p.249)

Fractions means division. *By dividing the numerator by the denominator, we change a common fraction to a decimal.* In general, a common fraction can be changed to either a terminating decimal or a repeating decimal.

A. Terminating Decimals

Common Fractions	Terminating Decimals	Decimal Fractions	Common Fractions
1/2	.5	5/10	1/2
1/4	.25	25/100	1/4
1/5	.2	2/10	1/2
3/4	.75	75/100	3/4

divide → equal → reduce

Clue: If the denominator of the fraction *divides evenly into 10 or 100,* the fraction can be changed easily to decimal or decimal fraction.

Example: 2, 5, divide evenly into 10, 100; 4 divides evenly into 100, because **2 and 5 are factors of 10, 100; 4 is a factor of 100.**

Common Fractions - Repeating Decimals (see p.250)

B. Repeating Decimals

				(a)		(b)		(c)
1/3	=	1 ÷ 3	=	.33...,	or	$.3\overline{3}$,	or	.33 1/3
2/3	=	2 ÷ 3	=	.66...,	or	$.6\overline{6}$,	or	.66 2/3
1/6	=	1 ÷ 6	=	.16...,	or	$.16\overline{6}$,	or	.16 2/3

In dividing the numerator by the denominator, sometimes the division never comes out evenly no matter how many times we divide. **If a digit or a group of digits repeats itself,** we call it a repeating decimal. A repeating decimals can be expressed in one of the following ways:

(a) By three dots - Indicates that the **last digit** repeats without end.
Read the three dots as *"and so on."*
(b) By a bar - Indicates that **the digit(s) under the bar** repeats endlessly.
(c) By a fraction - Write the remainder over the divisor.
Read .33 1/3 as *"thirty-three and one-third hundredths."*
(d) By rounding - Find the quotient **one more place than required** and round off.

Clue: If the denominator of the fraction *does not* **divide evenly into 10, 100,** the fraction will be a repeating decimal.

Estimating Fractions (Review first "Rounding" p.30)

Rounding is the method we use in estimating the sums and differences of fractions including mixed numbers. We compare the numerator and the denominator of the given fraction and round it either to 0, or 1/2, or 1 as described in the following:

Examples:	**Compare:**	**Rounded to:**
$\frac{1}{8}$, $\frac{1}{25}$, $\frac{3}{64}$ $\frac{5}{72}$,	The numerator is much smaller than the denominator.	**0**
$\frac{3}{8}$, $\frac{4}{9}$, $\frac{7}{13}$, $\frac{13}{25}$,	The numerator is about half of the denominator.	$\frac{1}{2}$
$\frac{5}{6}$, $\frac{7}{8}$, $\frac{10}{12}$, $\frac{28}{30}$,	The numerator is about the same as the denominator.	**1**

Fractions that are less than 1/2 are rounded down to 0 because it is closer to 0 on a number line.(See p.31)

Multiplying the numerator by 2, if the product is close to the denominator, round to 1/2.

Guideline: If a fraction is less than 1/2, round down to 0. If it is greater than 1/2, round up to 1.

Estimating Sums and Differences of Fractions and Mixed Numbers (Read p.230)

a) Estimate 3 1/7 + 5 8/9 + 4/5 ($\approx$ means "**about**" or "**is approximate to**")

$3 + 5 = 8$ **Step 1.** Add the whole number parts.

$\frac{1}{7} \approx 0 \quad \frac{8}{9} \approx 1 \quad \frac{4}{5} \approx 1$ **Step 2.** **Round** the fractional parts and estimate the sum: 2.

$8 + 2 = \mathbf{10}$ **Step 3.** Add the two sums: **about 10.**

b) Estimate 9 4/7 - 1 7/8 (The symbol   means "**therefore**")

$9 - 1 = 8$ **Step 1.** Subtract the whole number part.

Since 4/7 $\approx$ 1/2, 7/8 $\approx$ 1
4/7 < 7/8

$\therefore 9\ 4/7 - 1\ 7/8 < \mathbf{8}$

Step 2. **Adjust** the above difference by comparing the **fractional parts.** Since 4/7 is less than 7/8, the estimated difference will be **less than 8.**

Note: A **simpler way** to estimate the sums and differences of **mixed numbers** is to round the fractional part to the nearest whole number and then add or subtract.

Estimating Products of Fractions

We can estimate the products of fractions and mixed numbers either by **rounding** or by **using compatible numbers.**

a) Estimate 7 1/2 x 5 5/6

$$\begin{array}{rcr} 7\ 1/2 & \longrightarrow & 7 \\ \underline{x\ 5\ 5/6} & \longrightarrow & \underline{x\ \ 6} \\ & & 42 \end{array}$$

Using Rounding (Review first p.232)
1st. Round the fractional part in each factor to the nearest whole number.
2nd. Multiply the **rounded factors: about 42.**

b) Estimate 33 2/9 x 16 8/9

$$\begin{array}{cccccc} & 33\ 2/9 & x & 16\ 8/9 & & \\ & \downarrow & & \downarrow & & \\ & 30 & x & 20 & = & 600 \\ \text{or} & 30 & x & 15 & = & 450 \end{array}$$

Using Compatible Numbers
Substitute the given factors with a pair of numbers that are **close in value,** which you can **multiply mentally.** Often there are more than one pair of compatible numbers.

Note: The actual product is between 450 and 600. However, the first estimated product 600 is closer to the actual product because one factor was rounded down and the other factor rounded up.

Estimating Quotients of Fractions

Estimate the quotients of fractions in the same way you estimate the products of fractions.

a) Estimate 11 12/13 ÷ 3 7/9.

$$\begin{array}{ccccc} 11\ 12/13 & \div & 3\ 7/9 & & \\ \downarrow & & \downarrow & & \\ 12 & \div & 4 & = & 3 \end{array}$$

Using Rounding (See p.233)
Round the fraction to the nearest whole number and divide.

b) Estimate 19 1/7 - 4 3/11

$$\begin{array}{ccccc} 19\ 1/7 & \div & 4\ 3/11 & & \\ \downarrow & & \downarrow & & \\ 20 & \div & 4 & = & 5 \end{array}$$

Using Compatible Numbers
Since 19 **can not be divided** by 4, rounding is not the method to use. **Replace** both numbers with a pair of numbers, such as 5 x 4 = 20, **that can divide easily.**

Remember: When using compatible numbers, **choose the numbers that are close to the original numbers,** so that the estimated product or quotient will be closer to the actual product or quotient.

Prior Knowledge For Working With Fractions

Many students got lost in the maze of fractions because they didn't realize that fractions, unlike whole numbers and decimals, requires the **prior knowledge of many concepts and skills.** The check (√) in the following chart shows you the concepts and skills that are needed to do each operation:

Concepts & Skills	Addition	Subtraction	Multiplication	Division
* Multiplication and Division facts	√	√	√	√
* Finding LCM/LCD	√	√		
* "1" in fraction form	√	√	√	√
* Writing equivalent fractions	√	√	√	√
* Borrowing		√		

Note: Multiplication/Division facts are absolutely essential in working with fractions.

Concepts & Skills	Addition	Subtraction	Multiplication	Division
* Changing mixed numbers to improper fractions			✓	✓
* Writing whole numbers as fractions			✓	✓
* Writing the reciprocal of a number				✓
* Cancellation			✓	✓
* Changing improper fractions to mixed numbers	✓	✓	✓	✓
* Finding GCF	✓	✓	✓	✓
* Reduce fractions to lowest terms	✓	✓	✓	✓

Note: It is based on "1 property of multiplication/division" that we write equivalent fractions, do cancellation, etc.

Reviewing The Basic Concepts & Skills

* To Change A Mixed Number to An Improper Fraction (See p.95)

Remember: A mixed number = whole number + fraction (with "+" sign omitted)

(mixed number) — 5 1/3 $= \frac{(5 \times 3) + 1}{3} = \frac{16}{3}$ — (improper fraction)

$$\boxed{\frac{\textbf{(whole number x denominator) + numerator}}{\textbf{denominator}}}$$

* To Change An Improper Fraction to A Mixed Number (See p.94)

(improper fraction) $\frac{16}{3} = 16 \div 3 =$ 5 1/3 (mixed number)

$$\boxed{\textbf{numerator} \div \textbf{denominator}}$$

* To Write The Reciprocal Of a Number is to Invert the number (See p.326)

(fraction) $\frac{3}{5} \rightarrow \begin{matrix} 5 \\ 3 \end{matrix}$ (Reciprocal of 3/5)

(To invert means to switch the top and the bottom.)

* To write "1" as a fraction (See p.299)

$$1 = \frac{3}{3} = \frac{5}{5} = \frac{3/4}{3/4} \ldots$$

Use **the same number** for the numerator and the denominator.

* To Write A Whole Number as A Fraction (See p.298)

$$4 = \frac{4}{1}, \quad 7 = \frac{7}{1},$$

Use the whole number as the numerator and **1 as the denominator.**

* To Write An Equivalent Fraction of Higher Terms (See p.301)

$$\frac{2}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9}$$

Multiply the numerator and denominator of the fraction **by the same non-zero number. 3/3 = 1.**

* To Reduce A Fraction To Lowest Terms (See p.306)

$$\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$$

Divide the numerator and denominator of the fraction **by the largest number (GCF)** that ***divide evenly*** into both numbers.

Remember, fractions are always written in lowest terms.

Summary

(Introduction)

- Fractions means division. By dividing the numerator (top number) by the denominator (bottom number), we change a common fraction, or any fraction, to a decimal. If the denominator of a common fraction is a factor of 10 or 100, the decimal will be a terminating decimal, if it is not a factor of 10, or 100, the decimal will be a repeating decimal.
- A repeating decimal can be expressed (a) by three dots, (b) by a bar, (c) by a fraction, or (d) by rounding off to the desired place. Fraction form is the best way to express a repeating decimal.
- To avoid computational error, estimate the answer first. Then compare the answer with the original estimation.
- Any whole number can be written as a fraction by using the number as the numerator and "1" as the denominator To write "1" as a fraction, use any non-zero number for both the numerator and the denominator

Part IV. Fraction Operations

B. Factors & Multiples

Table of Contents

GCF (Greatest Common Factor) vs. LCM (Least Common Multiple)

Students often confuse GCF with LCM. Read the following lists side by side so that you see the differences between the two.

GCF has to do with:	**LCM** has to do with:
- Reducing a fraction to **lowest terms.**	- Changing **unlike** fractions to like fractions.
- **One** fraction.	- **Two or more** fractions
- The **numerator** and the **denominator** of a fraction.	- The **denominators** of unlike fractions
- Sums, differences, products, & quotients.	- Adding, subtracting, and comparing unlike fractions.
- Writing equivalent fraction **by division.**	- Writing equivalent fraction **by multiplication.**

GCF & Reducing A Fraction to Lowest Terms (Read first "GCF vs. LCM" p.280)

It is required that a fraction must be reduced to lowest terms at the end of computation. In order to reduce a fraction to lowest terms,

1st. You must know how to find the GCF of its numerator and denominator.

You will be able to find the GCF of two numbers,
a) if you know **what factors are** and how to find them. (pp.282,285)
b) if you know **factoring and prime factorization** (p.295)

2nd. You must know how to use the GCF to reduce the fraction.

You will be able to reduce a fraction,
a) if you know the "**1 property of division**" (p.298) and
b) if you know how to write "**1" in fraction** (p.299).

3rd. You must know when a fraction is in lowest terms.

You know when a fraction is in lowest terms,
a) if you know **prime and composite** numbers (p.290).
b) if you understand the meaning of "**relative prime**" (p.293).

Connection: It is important that you learn well factoring, prime factorization, & the concept of GCF. These skills are also used in Algebra to factor algebraic expressions, polynomials, quadratic equations, etc.

Factors - Factors & Divisors (See also "Divisibility Rules" p.288)

a factor		a factor		product
3	x	4	=	12

A number has a "limited" number of factors. And 1 is a factor of every number

$$\begin{array}{lr} \text{factor} & \\ \downarrow & 4 \\ 3 & \overline{)\,12} \\ & \underline{-\,12} \\ & 0 \end{array} \qquad \begin{array}{lr} \text{factor} & \\ \downarrow & 3 \\ 4 & \overline{)\,12} \\ & \underline{-\,12} \\ & 0 \end{array} \qquad \begin{array}{lrl} & 2 & \\ 5 & \overline{)\,12} & \\ & \underline{-\,10} & \\ & 2 & \leftarrow \text{remainder} \end{array}$$

Definition of Factors:

- Factors are numbers that multiply to give a product. We say, **"3 (or 4) is a factor of 12"** because 12 has 6 factors (1, 2, 3, 4, 6, 12) and 3 (or 4) is only one of them.

- Factors are ***exact divisors*** that divide evenly into a number. We say, **"3 (or 4) is *an exact divisor* of 12,** and also ***a factor*** **of 12"** because **division and multiplication are *inverse operations:***

12 - **3** (divisor) = **4** (quotient) **and** **4** (factor) x **3** (factor) = 12

This shows that we can use *division* to find the factors of a number.

Multiples - Multiples & Products (see also p.175)

$$\overset{\text{factor}}{3} \quad \text{x} \quad \overset{\text{factor}}{4} \quad = \quad \overset{\text{product}}{12}$$

multiples of 3

3 x 0 = 0
3 x 1 = 3
3 x 2 = 6
3 x 3 = 9
$\boxed{3 \text{ x } 4 = 12}$
.
.
.

multiples of 4

4 x 0 = 0
4 x 1 = 4
4 x 2 = 8
$\boxed{4 \text{ x } 3 = 12}$
.
.
.

A number has an "unlimited" number of multiples. And 0 is a multiple of every number

Definition of Multiples:

- A multiple of a number (say 3) is the product of that number (3) and any whole number (0, 1, 2, 3, ...). So, multiple is another name for product.

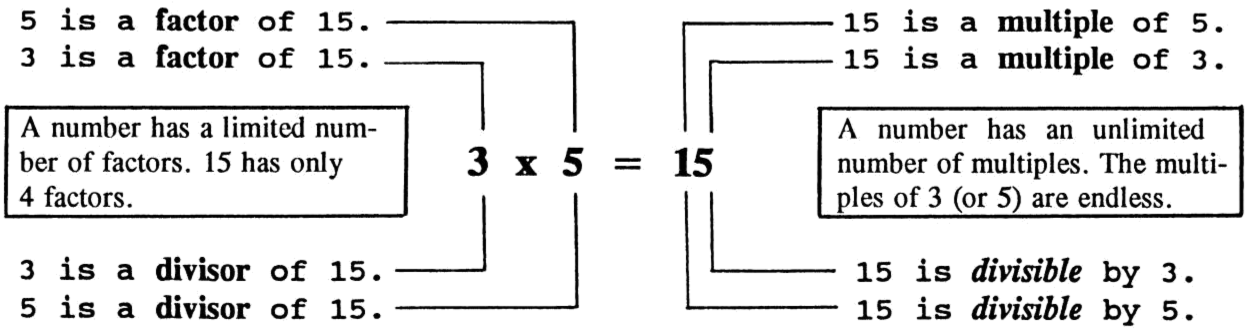
  We say, "12 is ***a*** multiple of 3 (or 4)" because 3 (or 4) has many multiples and 12 is ***one*** of them. And

  We say, "12 is ***a common multiple*** of 3 and 4" because some numbers have ***multiples in common.***

Remember: A product has to do with at least a pair of numbers - **two numbers;**
A product is a multiple of either number - **one number.**

Relating Factors, Divisors, Multiples, & Divisibility (Review first p 282,283)

The following example shows how Factors, Divisors, Multiples, and Divisibility are related. **A knowledge you need to have in working with fractions.**



5 is a **factor** of 15.

3 is a **factor** of 15.

A number has a limited number of factors. 15 has only 4 factors.

3 is a **divisor** of 15.

5 is a **divisor** of 15.

3 x 5 = 15

15 is a **multiple** of 5.

15 is a **multiple** of 3.

A number has an unlimited number of multiples. The multiples of 3 (or 5) are endless.

15 is ***divisible*** by 3.

15 is ***divisible*** by 5.

Remember the following statements:

a) **A number is a multiple of each of its factors/divisors.**

Example: 15 is a multiple of its factor/divisor.

b) **A number is divisible (can be divided) by another number if the second number is a factor or divisor of the first number.**

Example: 15 is divisible by 5 because 5 is a factor/divisor of 15.

Finding Factors: A How-To (Review first "Factors" p.282)

How To Find the Factors of 36:

$$\begin{aligned} 36 &= \overset{\text{factor}}{\overset{\downarrow}{1}} \text{ x } \overset{\text{factor}}{\overset{\downarrow}{36}} \\ &= 2 \text{ x } 18 \\ &= 3 \text{ x } 12 \\ &= 4 \text{ x } 9 \\ &= 6 \text{ x } 6 \end{aligned}$$

Method 1. Using Multiplication

Factors are numbers that you multiply together to give a product. List all the **pairs of numbers** when multiplied gives the product of 36. The factors of 36 are:

1, 2, 3, 4, 6, 9, 12, 18, 36

Remember:

* A number has a **limited** number of factors.

* **Each pair** of factors listed above is **a factor form** of 36.

* And 1 x 36, 2 x 18, 3 x 12, 4 x 9, 6 x 6, are **all** the factor forms of 36.

Note: We actually use **division** to find the pair of factors such as 1 x 36, 2 x 18, 3 x 12, because we remember only the one-digit multiplication facts.

Finding Factors: A How-To

How To Find the Factors of 36:

divisor quotient

36 ÷ 1 = 36
36 - 2 = 18
36 - 3 = 12
36 ÷ 4 = 9
36 ÷ 5 = 7 r1
36 ÷ 6 = 6
36 ÷ 7 = 5 r1
36 ÷ 8 = 4 r4
36 - 9 = 4

Method 2. Using Division

Divide 36 by 1, 2,... **the natural** numbers. The **exact divisors** and **their quotients** are the factors of 36 because division and multiplication are ***inverse operations:***

36 - 4 = 9, and
4 x 9 = 36

Therefore, the factors of 36 are:
1, 36, 2, 18, 3, 12, 4, 9, 6

Note: 5, 7, 8 are divisors but not factors of 36.

Note: We do not have to try any whole number greater than 6 as a divisor, since factors begin to repeat after 36 - 7 = 5 r1.

How to Find the Factors of 36:

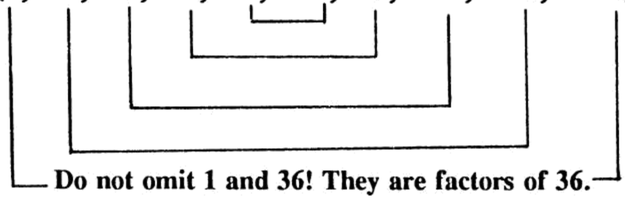
Method 3. This method is the same as **Method 2** except the way the factors are listed - from the smallest to the largest. Here is a how-to:

1st. Write 36 and a set of parentheses large enough to list the factors of 36: 36 ( ).

2nd. Start from the smallest divisor. Write it at the left end and its quotient at the other end: 36 (1, 36)

3rd. Then the next smallest divisor and its quotient. Continue in this manner until all factors of 36 are listed:

36 (1, 2, 3, 4, 6, 6, 9, 12, 18, 36)



Do not omit 1 and 36! They are factors of 36.

Finding Factors By Using Divisibility Rules (Review first p.282)

Divisibility or divisible means **"that a number can be *divided evenly* with *no remainder."*** It refers to **exact divisor** and **its quotient** because division is a common way of finding the factors of a number

Divisibility rules help us to know, by simple inspection without dividing, **whether a number can be divided evenly by a certain number.** They are useful especially in dealing with large numbers.

*** A number is divisible by 2, if its last digit is 2, 4, 6, 8, or 0.**
Example: 4, 36, 74, 308, 15790,... each number has a factor of 2.

Remember: Any number whose last digit is 2, 4, 6, 8, 0, is an **even number.** They are multiples of 2.

*** A number is divisible by 5, if its last digit is 0 or 5.**
Example: 5, 45, 95, 100,... each number has a factor of 5.

*** A number is divisible by 10, if its last digit is 0.**
Example: 10, 400, 2,000,... each number has a factor of 10.

- **A number is divisible by 4, if the <u>last 2 digits</u> of the number are zeros or is divisible by 4.**

  Example: 536: 36 ÷ 4 = 9, 536 is divisible by 4. 4 is a factor.

- **A number is divisible by 8, if the <u>last three digits</u> of the number are zeros or is divisible by 8.**

  Example: 5384: 384 ÷ 8 = 48; 5384 is divisible by 8. 8 is a factor.

- **A number is divisible by 3, if the <u>sum of the digits</u> of the number is divisible by 3.**

  Example: 705: 7 + 0 + 5 = 12; 12 ÷ 3 = 4
  705 is divisible by 3. 3 is a factor.

- **A number is divisible by 9, if the <u>sum of the digits</u> of the number is divisible by 9.**

  Example: 1697: 1 + 6 + 9 + 7 = 23; 23 ÷ 9 = 2 **R5.**
  1697 is not divisible by 9. 9 is not a factor.

- **A number is divisible by 6, if the number is divisible by 2 and also by 3.**

  Example: 1422: 2 ÷ 2 = 1 (divisible by 2)
  1422: 1 + 4 + 2 + 2 = 9; 9 ÷ 3 = 3 (divisible by 3)
  Therefore, 1422 is divisible by 6. 6 is a factor.

Connection: These rules are helpful in finding factors and also in factoring.

Factors - Prime & Composite Numbers (See also p.10)

When we examine the factors of say the first 11 natural numbers, we observe the characteristic of two groups of numbers as seen below:

Number	List of Factors	Number of Factors
1	1	1
2	1, 2	2
3	1, 3	2
4	1, 2, 4	3
5	1, 5	2
6	1, 2, 3, 6	4
7	1, 7	2
8	1, 2, 4, 8	4
9	1, 3, 9	3
10	1, 2, 5, 10	4
11	1, 11	2

Observations

* All numbers, except 1, have at least 2 factors.

* The numbers 2, 3, 5, 7, 11, have **only 2 factors** **1 and the number itself.** We call these numbers **Prime Numbers.**

* The numbers which have **more than 2 factors** are called **Composite Numbers.**

We do not know how many prime numbers there are because there is no way to predict when it will occur. However, we can use Eratosthenes' method to sort out the prime numbers up to certain numbers. See next page.

Finding the Prime Numbers (Compare with "Even & Odd Numbers" p.9)

	②	③	~~4~~	⑤	~~6~~	⑦	~~8~~	~~9~~	~~10~~
⑪	~~12~~	⑬	~~14~~	~~15~~	~~16~~	⑰	~~18~~	⑲	~~20~~
~~21~~	~~22~~	㉓	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	㉙	~~30~~
㉛	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	㊲	~~38~~	~~39~~	~~40~~
㊶	~~42~~	㊸	~~44~~	~~45~~	~~46~~	㊼	~~48~~	~~49~~	~~50~~

The Sieve of Eratosthenes

Around 200 B.C. Eratosthenes, a Greek mathematician, developed the following method to sort out the prime numbers.

First, write the whole numbers up to 50 or 100 and then follow the step described below:

Step 1. Circle 2. Then cross out all the multiples of 2 which are the numbers that can be divided evenly by 2.

Step 2. Circle 3, the next number not crossed out. Again, cross out all the multiples of 3. **Some numbers have multiples in common.**

Step 3. Circle 5, the next number not crossed out. Cross out all the multiples of 5.

Step 4. Continue the procedure until all numbers are either circled or crossed out. **The circled numbers are prime** and the crossed out numbers are composite.

Note: 1 is omitted because 1 is neither prime nor composite. It is a set by itself.

Prime Numbers vs. Composite Numbers (See also p.10)

The set of natural numbers is made up of three sets of numbers: the set of 1 by itself, the set of prime numbers, and the set of composite numbers.

Definition of Prime Number:

* A number is a prime number if it has **only two factors - 1 and itself.**

* A number is a prime number if it is **divisible only by 1 and itself.**

* A number is a prime number if it can be written **as a product** of two different numbers in **only one way, the order may differ.**
Example: 5 = 1 x 5 or 5 x 1 (Differ only in order)

Definition of Composite Number:

* A number is a composite number if it has other **factor(s) in addition to 1 and itself.**

* A number is a composite number if it is **divisible by a natural number other than 1 and itself.**

* A number is a composite number if it can be written **as a product** of two different natural numbers in **more than one way.**
Example: 12 = 1 x 12 & 12 = 2 x 6 & 12 = 3 x 4

Relatively Prime Numbers And Lowest Terms

Relatively Prime Numbers:

Two numbers are said to be relatively primes if **1 is the only common factor.** The GCF or the GCD of the numbers is 1. For example:

4 and 9 are relatively prime.
The factors of 4: **1**, 2, 4
The factors of 9: **1**, 3, 9

The only common factor of 4 and 9 is **1**.

Two numbers are relatively prime even if one or both are composite like 4 and 9; 15 and 16.

Lowest Terms or Simplest Form:

A fraction is said to be in lowest terms **when its numerator and denominator are relatively prime**, which means when their GCF is 1. For example:

4/9 is in lowest terms since 4 and 9 are relatively prime.

To reduce a fraction to lowest terms, we divide its numerator and denominator by the greatest common factor (GCF). That is why GCF is also called the greatest common divisor (GCD).

Factor & Factoring (Review first "Factors" p 282)

Factor(s):

$$\overset{\text{factor}}{\overset{|}{3}} \times \overset{\text{factor}}{\overset{|}{6}} = 18$$

* Factor as a noun refers to **the numbers** when multiplied give the product. The factors of 18 are **1, 2, 3, 6, 9, 18.**

Factor, Factoring:

$$\begin{aligned} 18 &= 3 \times 6 \\ &= 2 \times 9 \\ &= 1 \times 18 \end{aligned}$$

Note: 2 & 3 are prime factors ; 6, 9, 18 are composite factors

* Factor as a verb refers to **the process** of writing a number as a product of any two of its factors, prime or composite. We say:

- **"18 has been factored."** And
- **"3 x 6 is *a* factor form of 18"** **because 18 has other factor forms.**

Connection: Factoring is an important process in mathematics, especially in Algebra.

Prime Factoring, Prime Factorization (Review "Prime & Composite" p.292)

Prime Numbers:

2, 3, 5, 7, 11

* Prime refers to **the numbers** that have **only two factors**. Memorize the **first 5 prime numbers** on the left.

Prime Factoring, Prime Factorization:

factoring prime factoring

$$18 = \underline{3 \text{ x } 6} = \boxed{3 \text{ x } 2 \text{ x } 3} \quad ①$$

$$18 = \underline{2 \text{ x } 9} = \boxed{2 \text{ x } 3 \text{ x } 3} \quad ②$$

* Prime factoring is **the process** of writing a **composite** number as the product of **prime** factors.

* Prime factorization is **the result** of prime factoring. We say, **"18 has been factored into prime numbers."** **"3 x 3 x 2 is the prime factorization of 18."**

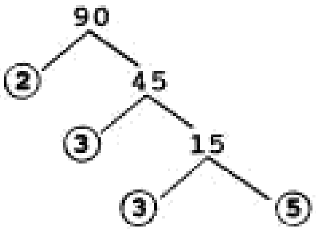
Note: Each composite number has **"only one set of prime factors"** – **the factors may be in different order** as in ① & ② above. We can factor any composite number into prime factors either by **"Factor Tree"** or by **"Invert Short Division"** **(See next pages).**

Prime Factorization: Factor Tree Method (Review first "Prime & Composite" p.292)

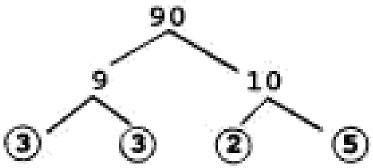
To Factor 90 into Prime Numbers by Using The Factor Tree:

Write 90 as the **product of two of its factors.** Write one on the left and the other on the right under the number 90. Continue to write **composite** numbers as the products of two factors until **all factors are prime.**

a)



b)



Note: The prime factors of 90 are the same for both a) and b).

The prime factorization of 90 is 2 x 3 x 3 x 5.

The **Fundamental Theorem of Arithmetic** states that every composite number has *exactly one prime factorization* though the order of the factors may differ.

Prime Factorization: Invert Short Division Method (See also "GCF Method" p.306)

Prime factorization is one of the methods used to find the GCF of two (or more) numbers. GCF is used in reducing a fraction to lowest terms.

To Find the GCF of 54 and 90 by Using Prime Factorization:

Step 1. Factor each number into prime factors. Divide each number by **its own prime** factor starting with *smallest prime number.* Continue the process until the **last number (quotient) is prime.**

prime →	2	) 54	prime →	2	) 90
prime →	3	) 27	prime →	3	) 45
prime →	3	) 9	prime →	3	) 15
prime →		3	prime →		5

Step 2. Write each number as the product of prime factors.

The prime factorization of 54: 2 x 3 x 3 x 3

The prime factorization of 90: 2 x 3 x 3 x 5

Step 3. Find the GCF of two numbers. The GCF is the product of the factors which appear in both prime factorizations.

GCF of 54 and 90: 2 x 3 x 3 = 18

"1" Property of Multiplication & division (Review first p.121)

It is based on the following two properties that we write all equivalent fractions. Read this page and next page together.

"1" property of multiplication:

8 x 1 = 8 The property says, **"Any number multiplied by 1 is the number."**

3/7 x 1 = 3/7 The property applies also to fractions.

"1" Property of Division:

15 ÷ 1 = 15 The property says, **"Any number divided by 1 is the number."**

8/9 ÷ 1 = 8/9 Again it applies also to fractions.

Writing Whole Numbers As Fractions - Any whole number can be changed to a fraction by **using the number as the numerator** and **1 as the denominator.**

Example: 5 = 5/1; 12 = 12/1, etc.

Writing "1" in Fraction Form

To use "1" properties (see opposite page) in fractions, we write 1 as improper fractions **with the same number for both numerator and denominator (0 excluded)** because **"Any number divided by itself equals 1"** (p.196) That means we can write 1 as fractions in *many ways* - either as **simple fractions** or as **complex fractions.**

*** Write 1 As Simple Fractions** (See p.92)

$$1 = \frac{2}{2} = \frac{25}{25} = \frac{100}{100}, \ldots$$

To multiply/divide a fraction by 2/2, **is the same as** to multiply/divide it by 1. **The value of the fraction is not changed.**

Note: This knowledge is needed in subtracting fractions with borrowing.

*** Write 1 As Complex Fractions**

$$1 = \frac{\frac{2}{3}}{\frac{2}{3}} = \frac{\frac{5}{1}}{\frac{5}{1}} = \frac{\frac{9}{4}}{\frac{9}{4}}, \ldots$$

By applying "1" property of multiplication with 1 written as complex fraction, we change division of fraction to multiplication (See p.327, 365)

Connection: "1" properties with 1 written in fractions are used in writing **equivalent fractions** (p.300), in **cancellation** (p.304), etc.

Writing Equivalent Fractions : A How-To

By using the **"1" property of multiplication/division with** *1 written as fractions* (Review also last two pages), we can write fractions of **equal value** to replace the given fraction. Here is how we do it:

To Write Equivalent Fraction With Lower Terms:

1 in fraction

6/9, 4/6, 2/3 are all equivalent fractions of 12/18

$$\frac{12}{18} = \frac{12}{18} \div \frac{2}{2} = \frac{12 \div 2}{18 \div 2} = \frac{6}{9}$$

$$\frac{12}{18} = \frac{12}{18} - \frac{3}{3} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}$$

$$\frac{12}{18} = \frac{12}{18} - \frac{6}{6} = \frac{12 - 6}{18 - 6} = \frac{2}{3}$$

To write equivalent fraction with lower terms, ***divide*** the numerator and the denominator of a given fraction by their common factors.

To divide the fraction, 12/18, by 2/2, 3/3 etc. is the same as to divide the fraction by 1.

Connection: ***By dividing*** a fraction with the GCF, we reduce the fraction to lowest terms.

To Write Equivalent Fraction With Higher Terms:

1 in fraction

4/6, 6/9, 10/15,... are equivalent fractions of 2/3.

$$\begin{aligned}\frac{2}{3} &= \frac{2}{3} \times \frac{2}{2} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6}\\ &= \frac{2}{3} \times \frac{3}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9}\\ &= \frac{2}{3} \times \frac{5}{5} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}\\ &\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots\end{aligned}$$

To write equivalent fractions with higher terms, ***multiply*** the numerator and the denominator of the given (original) fraction with the same **non-zero number:** 2/2, 3/3, etc.

To multiply the fraction by 2/2, etc. is the same as to multiply it by 1.

Remember: In general, we can ***multiply*** the numerator and the denominator of a fraction **by any non-zero number;** but they can be ***divided only*** **by common factors.**

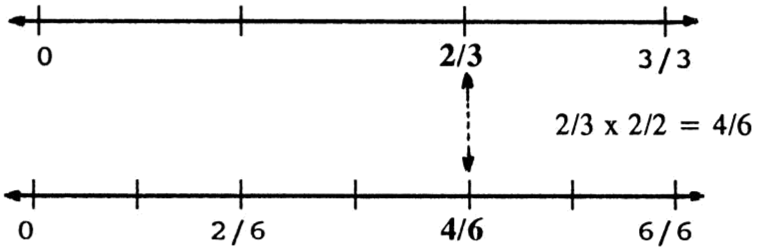
Connection: *By multiplying,* we change fractions of different denominators to fractions of the same denominator using the LCM. **(See p.311)**

Test Of Equivalent Fractions

To write equivalent fractions, we **either multiply or divide** the given fraction by **a fraction which equals to 1** -- 2/2, 3/3, 4/4,... (p.300)
To test whether or not two fractions are equivalent, we can use one of the following methods:

1. Use A Number Line

If two fractions are equal, they will have the **same point** on the number line. Let's locate the point for 2/3 and 4/6 on the number lines to see if they are equivalent fractions:



Divide 1 into 3 equal parts, and locate 2/3 on the line.

Divide 1 into 6 equal parts, and locate 4/6 on the line.

We know that 2/3 and 4/6 **are equivalent fractions** because they share the same **point** on the number line.

2. Cross Products or Cross Multiplication (See also "Proportion" p.375)

To determine whether or not 3/7 and 12/28 are equivalent:

1st. Multiply the denominator of the first fraction by the numerator of the second:

$$7 \times 12 = \mathbf{84}$$

2nd. Multiply the numerator of the first fraction by the denominator of the second:

$$3 \times 28 = \mathbf{84}$$

$$\frac{3}{7} = \frac{12}{28}$$

3rd. If the product of the first one ***equals*** the product of the second one, the two fractions are equivalent: 84 = 84.

Note: Do you see why the method is called "cross products"?

3. Compare Fractions in Lowest Terms

To decide whether or not 14/21 & 24/40 are equivalent:

$$\frac{14}{21} = \frac{14 - 7}{21 \div 7} = \frac{2}{3}$$

$$\frac{24}{40} = \frac{24 - 8}{40 \div 8} = \frac{3}{5}$$

1st. Reduce each fraction to lowest terms using **any method** given on pages 306-309.

2nd. Compare the fractions in lowest terms to see if they are identical. If not, the two fractions are not equivalent.

Cancellation - Cancelling Factors

To cancel means ***to divide*** **(not subtract).** Cancellation is used to simplify computation by **eliminating common factors** before multiplying. It utilizes the **"1" property of division** - the concept that lies behind the following computation:

1. In Multiplication of Fractions: 27/12 x 2/9 x 4/3

$$\frac{\overset{\overset{1}{\cancel{3}}}{\cancel{27}} \times \overset{1}{\cancel{2}} \times \overset{2}{\cancel{4}}}{\underset{\underset{3}{\cancel{6}}}{\cancel{12}} \times \underset{1}{\cancel{9}} \times \underset{1}{\cancel{3}}} = \frac{2}{3}$$

① Divide 12 & 2 by 2 → 6 & 1
② Divide 6 & 4 by 2 → 3 & 2
③ Divide 27 & 9 by 9 → 3 & 1
④ Divide 3 & 3 by 3 → 1 & 1

You can cancel factors in any order, as long as the numerator and the denominator has a common factor Knowing the "Divisibility Rules" would help you to find factors (p.288-289)

2. In Reciprocals: 5/7 x 7/5 = 1

$$\frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{7}}} \times \frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{5}}} = 1$$

or

$$\frac{5}{7} \times \frac{7}{5} = \frac{35}{35} = 1$$

(The fraction 35/35 = 1)

Use either cancellation or multiplication, the product of the reciprocals is always 1.

3. In Reducing a Fraction to Lowest Terms: 24/54

$$\frac{24}{54} \underset{\div 2}{\overset{\div 2}{=}} \frac{12}{27} \underset{\div 3}{\overset{\div 3}{=}} \frac{4}{9} \quad \text{or} \quad \frac{24 = 2 x 3 x 4}{54 = 2 x 3 x 9} = \frac{4}{9}$$

Note: "Successive Deduction" (p.308) and "Prime Factorization" (p.309) utilize cancellation.

4. In Division of Whole Numbers: Divide 720 by 48

$$\frac{720 = \cancel{2} x 3 x \cancel{4} x 5 x \cancel{6}}{48 = \cancel{2} x \cancel{4} x \cancel{6}} = 3 x 5 = 15$$

The answer is 15.

Write as a fraction. Factored and then cancel. Using long division gives the same answer

Remember:

$\frac{3 x \cancel{7}}{\cancel{7}}$	**You can cancel the 7's, the common factor, because multiplication (3 x 7) and division (1/7) are inverse operations.**
$\frac{3 + 7}{7}$	**You can not cancel the 7's because 7 is not a common factor Addition (3 + 7) and division (1/7) are not inverse operations.**
$\frac{37}{7}$	**You can not cancel the 7 in numerator and the 7 in denominator, because they are digits, not factors.**

Reducing A Fraction To Lowest Terms: GCF Method

To Reduce 24/36 by Common Factors, GCF:

Step 1. List **all** the factors of 24 & 36. **(See Method 3 on p.287)**

24 (**1**, **2**, **3**, **4**, **6**, 8, **12**, 24)
36 (**1**, **2**, **3**, **4**, **6**, 6, 9, **12**, 18, 36)

Step 2. List the factors that are **common** to both lists.

The common factors of 24 and 36 are:
1, 2, 3, 4, 6, **12**

Step 3. Choose the **greatest common factor** (GCF), the **largest number** found in the above list of the common factors.

The GCF of 24 and 36 is **12.**

Step 4. **Divide** 24 & 36 by 12, their GCF, respectively.

$$\frac{24 \div 12}{36 \div 12} = \frac{2}{3}$$ The lowest terms of 24/36.

Note: See comment on this method on p.310.

Reducing A Fraction To Lowest Terms: Invert Short Division Method

To Reduce 48/72 by Invert Short Division:

Step 1. Write 48 & 72 side by side as below, **the numerator first.** Divide both numbers by **any common factor.** Continue the process until the numbers are **relatively prime.**

common factor

composite

$$\begin{array}{r|rr} 2 & 48 & 72 \\ \hline 2 & 24 & 36 \\ \hline 6 & 12 & 18 \\ \hline & 2 & 3 \end{array}$$

2 & 3 are relative prime.

48/72

2/3 ← **lowest terms**

Step 2. **Multiply the common factors** to get GCF of 48 & 72.
The GCF/GCD of 48 & 72: **2 x 2 x 6 = 24**

Step 3. **Divide** 48/72 by their GCF 24 respectively.

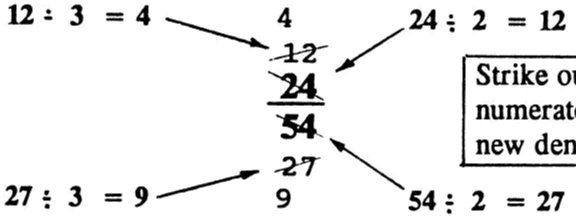
$$\frac{48 \div 24}{72 \div 24} = \frac{2}{3}$$ 2/3, the lowest terms of 48/72.

Note: To reduce a fraction to lowest terms, Steps 1 is all you need.

Reducing A Fraction To Lowest Terms: Successive Deduction Method

To Reduce 24/54 by Successive Deduction:

Divide the numerator and the denominator by **obvious common factor.** Continue to divide the new numerator and denominator by common factor until they are **relatively prime.**



Strike out 24 & 54. Write 12, the new numerator, above 24 and 27, the new denominator, below 54. Repeat.

Each time you divide 24/54 by a common factor, you are writing **an equivalent fraction** of 24/54 in lower terms:

$$\frac{24}{54} = \frac{12}{27} = \frac{4}{9}$$

So, The lowest terms of 24/54 is 4/9. (4 & 9 are relatively primes.)

Note: This method is similar to Invert Short Division.

Reducing A Fraction to Lowest Terms: Prime Factorization Method (see p 295)

To Reduce 48/54 by Prime Factorization:

Step 1. Factor each number into primes numbers. Methods given on p 297

Step 2. Replace each number with its prime factorization.

$$\frac{48}{54} = \frac{2 \times 2 \times 2 \times 2 \times 3}{2 \times 3 \times 3 \times 3}$$

Step 3. Cancel all the factors common to both numbers. (see p.304)

$$\frac{48}{54} = \frac{\cancel{2} \times 2 \times 2 \times 2 \times \cancel{3}}{\cancel{2} \times 3 \times 3 \times \cancel{3}} \quad \text{or} \quad \frac{2}{2} \times \frac{3}{3} = \frac{2 \times 2 \times 2}{3 \times 3}$$

Or you may arrange the factors as a fraction equals 1.
The GCF of 48 & 54 is the product of their common factors: 2 x 3 = 6.

Step 4. Multiply the remaining factors in the numerator to find the new numerator, and the remaining factors in the denominator to get the new denominator.

$$\frac{48}{54} = \frac{2 \times 2 \times 2}{3 \times 3} = \frac{8}{9}$$

8 & 9 are relatively prime.
Their GCF is 1

Reducing A Fraction To Lowest Terms: Comments on the Methods

1. **Common Factors Method** (p.306) **vs. Prime Factorization Method** (p.309)
   - * Finding the GCF of two numbers by "Common Factors Method" is cumbersome and impractical when the numerator and the denominator are large numbers.
   - * Finding the GCF of two numbers by Prime factorization is simpler than "Common Factors Method."

2. **Invert Short Division Method** (p.307) **vs. Prime Factorization Method** (p.309)
   - * In regular Invert Short Division Method, you **divide both numbers by any common factor - prime or composite.** To write a fraction in lowest terms, the steps of finding the GCF and dividing the fraction by their GCF can be omitted.
   - * In Prime Factorization-Invert Short Division, you **divide each number individually by its prime factor(s) only.** To write a fraction in lowest terms, the steps of finding the GCF and divide the fraction by their GCF can not be omitted.

3. **Successive Deduction Method** (p.308) **vs. Invert Short Division Method** (p.307)
   - * Successive Deduction Method is in fact similar to Invert Short Division Method. However, it is the only method that does not require to find the GCF of the numerator and the denominator.

LCM & Changing Unlike Fractions To Like Fractions (Read "GCF vs. LCM" p.280)

Keep in mind that fractions can be **added, subtracted,** or **compared** only if they are like fractions (having the same number for the denominators). Changing unlike fractions (having different numbers for the denominators) to like fractions is a two-step process:

1st. You must know how to find the LCM of two or more numbers.

You will be able to find the LCM of two or more numbers,

a) if you know **what multiples are**, and **how to find them** (pp 283,312)

b) if you know **primes** and **prime factorization** (p.295)

c) if you know **exponents** (p.38).

2nd. You must know how to change unlike fractions to like fractions.

You will be able to change unlike fractions to like fractions,

a) if you know **"1 property of multiplication"** (p.298).

b) if you know how to write **"1" in fraction form** (p.299).

c) if you know how to write **equivalent fractions using the LCM** (p.320).

Connection: The concept of multiples and LCM is used in Algebra to solve the system of equations by elimination.

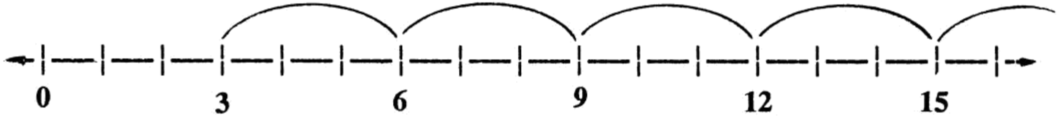
Finding Multiples: A How-To (Read first "Multiples" p.283)

How To Find the Multiples of 3:

To find the multiples of a number, **multiply the number by 0, 1, 2, 3,... the consecutive whole numbers.** The products are the multiples of the number. The following shows how to find the multiples of 3:

	3x0	3x1	3x2	3x3	3x4...	**multiply 3 by 0, 1, 2, 3,...**
Multiples of 3:	**0**	**3**	**6**	**9**	**12...**	**products are multiples of 3**

Another way to find the multiples of a number is **to count by the number (skip count).** To find the multiples of 3, count by 3s which can be shown on a number line.



Count by 3s: **3** **6** **9** **12** **15 ...** **multiples of 3**

+3 +3 +3 +3 **add 3**

Observe the pattern in the sequence of the multiples of 3. The rule for the sequence is adding 3.

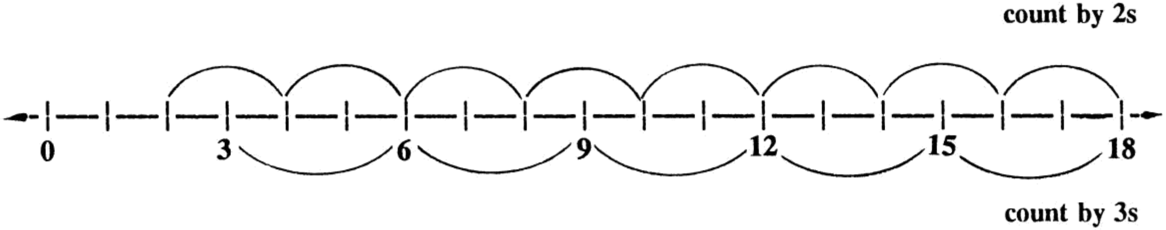
Finding Least Common Multiple (LCM): Method 1

Finding the LCM of 2 and 3 by Using The Number Line:

Since the number line can be used to find the multiples of a number **(see p.312)**, it can also be used to find the least common multiples (LCM) of two numbers. Here is a how-to:

1st, Draw the number line.

2nd, Show the **multiples of 2** on the number line - **above.**



3rd, Show the **multiples of 3** on the number line - **underneath.**

4th, The **common multiples** of 2 and 3 are the numbers(s) **where the multiples of 2 and 3 meet on the number line: 6, 12, 18,...**

5th, The LCM of 2 and 3 is **the first non-zero number where the multiples of two numbers meet: 6**

Finding Least Common Multiple (LCM): Method 2 (Review first p.312)

Finding the LCM of 4 and 6 by Common Multiples Method:

Step 1. List the first few multiples of the smaller number. Then list the multiples of the larger number.

multiples of 4: 0, 4, 8, **12,** 16, 20, **24,** ...
multiples of 6: 0, 6, **12,** 18, **24,** 30, 36, ...

If no common multiple is found, list several additional multiples of the smaller number Then list multiples of the greater number until a common multiple is found. That multiple is the LCM. In that case, Step 2 & 3 are not needed.

Step 2. Identify the multiples which are **common to both numbers.**

The common multiples of 4 and 6 are 12, 24,... (the multiples of 12)

If zero is included in the list of common multiples, then the LCM will always be 0 because 0 is the first multiple of every number

Step 3. Choose the Least Common Multiples (LCM). The LCM is the **smallest non-zero number** common to both numbers.

The LCM of 3 and 4 is 12.

Finding Least Common Multiple (LCM): Method 3 (Review first P 312)

Finding the LCM of 4 & 15 by the Multiples of Larger Number:

First, list the first few multiples of the larger number of the two numbers. Then, **test each multiple to see whether it is divisible by the smaller number.** The first multiple of the larger number that can be divided by the smaller number evenly is the LCM.

The multiples of 15:	0,	15,	30,	45,	60,...
Is the multiple divisible by 4?:	No	No	No	No	Yes

So, the common multiples of 4 and 15 are 60, 120, 180,... (the multiples of 60)

The LCM of 4 and 15 is 60 or The LCM(4, 15) = 60

General Rules:

- * **If two numbers are relatively prime** like 4 and 15 (which means their GCF is 1), the LCM of the two number is **their product:** 4 x 15 = 60.
- * **If the larger number is a multiple of (or divisible by) the smaller number** like 4 and 32, **the LCM is the larger number of the two:** 32 (32 ÷ 4 = 8)

Finding Least Common Multiples (LCM): Method 4

Finding the LCM of 18, 48, & 60 by Invert Short Division:

Step 1. Divide the numbers (dividends) ***by any prime number*** **that will divide evenly into most of them. Bring down the number *which is not divisible by the prime*.** Continue the process **until the quotients are all primes.**

$$\begin{array}{r|rrr} 2 & 18 & 48 & 60 \\ \hline 2 & 9 & 24 & 30 \\ \hline 3 & 9 & 12 & 15 \\ \hline 2 & 3 & 4 & 5 \\ \hline & 3 & 2 & 5 \end{array}$$

(9 is not divisible by 2, bring 9 down.)

(3 & 5 are not divisible by 2, bring them down.)

All the divisors and quotients are prime numbers.

Step 2. To find the LCM of 18, 48, and 60, **multiply all the divisors and all the quotients.**

The LCM = $2 \times 2 \times 3 \times 2 \times 3 \times 2 \times 5 = 2^4 \times 3^2 \times 5 = 720$

Note: "Invert Short Division" (p.297) is one of the two methods used in prime factorization. The other is by **"Factor Tree".** (p.296)

Finding Least Common Multiple (LCM): Method 5

Finding the LCM of 18, 48, & 60 by Prime Factorization Method:

Step 1. Factor each number into prime by "Factor Tree Method" described on page 296. It is omitted here to save the space. (With some practice, you can factor a number into primes mentally.)

Step 2. Write each number as the product of prime factors. Then write the identical factors in **exponent.** (See p.38)

$18 = 2 \times 3 \times 3 = 2 \times \textcircled{3^2}$
$48 = 2 \times 2 \times 2 \times 2 \times 3 = \textcircled{2^4} \times 3$
$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times \textcircled{5}$

All the different factors are: 2, 3, 5

Step 3. To find the LCM, multiply together the highest powers of each of the different factors that occur in all three prime factorizations:

- the highest powers of 2 is 4 -- 2^4 (from 48)
- the highest powers of 3 is 2 -- 3^2 (from 18)
- the highest powers of 5 is 1 -- 5 (from 60)

Therefore, the LCM (18, 48, 60) $= 2^4 \times 3^2 \times 5 = 720$

Finding Least Common Multiple (LCM): Comments on the Methods

1. **The Number Line Method** (p.313)
   This method is given to show the beginners that multiples and counting by a number (skip count) are related.

2. **The Common Multiples Method** (p.314)
   This method is cumbersome when the numbers are large or when it involves more than two numbers.

3. **The Multiples of Larger Number Method** (p.315)
   This method is simple. All you need to do is write down the multiples of the larger number and the rest can be done mentally.

4. **Prime Factorization Method** (p.317) **& Invert Short Division** (p.316)
   Prime factorization, either by factor tree or by invert short division, is an efficient method. It is the only method which can be used to find the GCF as well as the LCM as you see on page 319.

   Note: If you have mastered the **division,** know the **divisibility rules,** and remember the **prime numbers,** you can factor a number into prime *mentally* You do all the divisions in your head and write down the prime factors.

Prime Factorization: LCM vs. GCF

(Review first "Exponents" p.38)

The following shows you how to use prime factorization to find the LCM and the GCF of 36 and 40. To avoid confusion, they are placed side by side so that you may see the similarity and the difference.

Finding the LCM

Step 1.

Factor each number into primes.

$36 = 2 \times 2 \times 3 \times 3$

$40 = 2 \times 2 \times 2 \times 5$

Write factors in exponent form.

$36 = 2^2 \times 3^2$

$40 = 2^3 \times 5$

Step 2.

To find the LCM of 36 and 40, **multiply the largest exponents of each of the different factors:**

$2^3 \times 3^2 \times 5 = 360$

The LCM (36, 40) = 360

Finding the GCF

Step 1.

Factor each number into primes.

$36 = 2 \times 2 \times 3 \times 3$

$40 = 2 \times 2 \times 2 \times 5$

Step 2.

To find the GCF of 36 and 40, **multiply the factors that are common to both numbers:**

$2 \times 2 = 4$

The GCF (36, 40) = 4

Changing Unlike Fractions to Like Fractions: A How-To (1)

Changing 5/18 and 7/12 to Like Fractions Using the LCM:

Step 1. Find the LCM of 18 and 12. The LCM of 18 & 12 is 36.
The LCM becomes the common denominator (LCD) of 5/18 & 7/12.

Step 2. Write equivalent fractions of 5/18 and 7/12 with 36 as the denominator.
Find the number by which you multiply the denominators 18 and 12 respectively to get 36.

$$\frac{5}{18} \underset{\times\ n}{=} \frac{?}{36} \qquad \frac{7}{12} \underset{\times\ m}{=} \frac{?}{36}$$

$n = 36 \div 18 = 2$

$m = 36 \div 12 = 3$

Then, Multiply 5/18 by 2/2 (= 1) and 7/12 by 3/3 (= 1).

$$\frac{5}{18} \overset{\times\ 2}{\underset{\times\ 2}{=}} \frac{10}{36} \qquad \frac{7}{12} \overset{\times\ 3}{\underset{\times\ 3}{=}} \frac{21}{36}$$

$$\frac{5}{18} \times \frac{2}{2} = \frac{10}{36} \qquad \frac{7}{12} \times \frac{3}{3} = \frac{21}{36}$$

Changing Unlike Fractions to Like Fractions: A How-To (2) (Review first p.303)

Changing 5/18 and 7/12 to Like Fractions By Cross-Multiplying:

Step 1. Same as in Method 1, see last page.

Step 2. Write equivalent fractions with 36 as the denominator. **Since we know the new denominator, we can use cross multiplication to find the new numerator. Let n represent the unknown numerator.**

Cross Multiplication

$$\frac{5}{18} = \frac{n}{36} \qquad (18 \times n = 5 \times 36,\ n = 10) \qquad \frac{5}{18} = \frac{10}{36}$$

$$\boxed{n = \frac{5 \times \overset{2}{\cancel{36}}}{\underset{1}{\cancel{18}}} = 10} \qquad \boxed{n = \frac{7 \times \overset{3}{\cancel{36}}}{\underset{1}{\cancel{12}}} = 21}$$

$$\frac{7}{12} = \frac{n}{36} \qquad (12 \times n = 7 \times 36,\ n = 21) \qquad \frac{7}{12} = \frac{21}{36}$$

So, the like fractions of 5/18 and 7/12 are **10/36 and 21/36.**

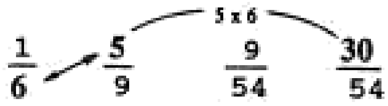
Changing Unlike Fractions to Like Fractions: A How-To (3)

Changing 1/6 and 5/9 by Multiplying the Given Denominators:

Step 1. To find the common denominator, multiply the given denominators:
6 x 9 = 54

Step 2. To find the numerator of the first fraction, multiply its numerator by the other denominator
1 x 9 = 9

Step 3. To find the numerator of the second fraction, multiply its numerator by the first denominator:
5 x 6 = 30



$$\frac{1}{6} + \frac{5}{9} = \frac{1 \times 9}{6 \times 9} + \frac{5 \times 6}{9 \times 6} = \frac{(1 \times 9) + (5 \times 6)}{(9 \times 6)}$$

It looks like this using addition.

So, the like fractions of 1/6 and 5/9 are **9/54 and 30/54.**

Changing Unlike Fractions to Like Fractions: A How-To (4)

Changing 1/2, 3/4 & 7/16 to Like Fractions: Special Case

General Rule: ***If one denominator*** (the larger number of the two or the largest number of three) ***is a multiple of the other denominator(s)***, the LCD is that number. **(See also page 315)**

First, ***examine*** **the denominators** 2, 4, 16. You notice that they are multiples of 2 and 16 is a multiple of 2 and 4:

16 is a multiple of 2 because 2 x **8** = 16.
16 is a multiple of 4 because 4 x **4** = 16.
The LCD of the equivalent fractions is **16.**

Next, ***write equal fractions*** of 1/2 and 3/4 **with 16 as the denominator.**

Multiply 1/2 by 8/8 (= 1) = 8/16 and
Multiply 3/4 by 4/4 (= 1) = 12/16.

So, the like fractions are **8/16, 12/16, & 7/16.**

Changing Unlike Fractions to Like Fractions: A How-To (5)

Changing 1/2, 4/7 and 2/9 to Like Fractions: Special Case

General Rule: ***If the denominators*** of two or more fractions are ***relatively prime*** which means their GCF is 1, the LCD of the fractions is **the product of the denominators.**

First, ***examine*** **the denominators** 2, 7, 9. You notice that they are relatively prime because their only common factor is 1.
So, the LCD of the fractions is **2 x 7 x 9 = 126**

Next, ***write equivalent fractions*** **with 126 as the denominator.**
126 ÷ 2 = **63** so, **multiply** 1/2 by **63/63** = 63/126
126 - 7 = **18** so, **multiply** 4/7 by **18/18** = 72/126
126 - 9 = **14** so, **multiply** 2/9 by **14/14** = 28/126

Shortcut: If 2 x 7 x 9 = 126 then, 126 ÷ 2 = 7 x 9 = 63
126 -- 7 = 2 x 9 = 18
126 ÷ 9 = 2 x 7 = 14

So, the like fractions are **63/126, 72/126, 28/126.**

Changing Unlike Fractions to Like Fractions: Comments on the Methods

Review first "LCM & Changing Unlike Fractions to Like Fractions" (p.311) and "Finding LCM" (p.317) because they are all related.

1. **The LCM Method** (p.320) **& The Cross-Multiplying Method** (p.321)
   These two methods are the same in using the LCM. But they differ in step 2 as to how to write equivalent fractions. Both methods are also used in "Ratios & Proportions".

2. **Multiplying the Given Denominators Method** (p.322)
   In case you forget how to find the LCM/LCD you will still be able to change unlike fractions to like fractions if you know this method. But this method could result in computing larger numbers than it is necessary. **For example,** the common denominator of 1/6 and 5/9 is **18** by using the LCM method, and **54, 3 times larger,** by using this method.

3. **Two Special Cases** (p.323) and (p.324)
   You would be able to use these two special cases, if you know the concepts of multiples, the rules of divisibility (p.288), and have memorized some of the prime numbers (p.291).

Finding The Reciprocal of A Number: A How-To

* To Find the Reciprocal of A Whole Number 5:

$$5 \longrightarrow \frac{5}{1}$$

1st. Write 5 as a fraction **with the denominator 1.**

$$\begin{tikzcd}[row sep=tiny] \underline{5} \arrow[rd] & \underline{1} \\ 1 \arrow[ru] & 5 \end{tikzcd}$$

2nd. To find the reciprocal, ***invert*** **the fraction** –
Change the places of the numerator and the denominator

* To Find the Reciprocal of A Mixed Number 4 3/8:

$$4\frac{3}{8} \longrightarrow \frac{35}{8}$$

1st. Change the mixed number to **improper fraction.**

$$\begin{tikzcd}[row sep=tiny] \underline{35} \arrow[rd] & \underline{8} \\ 8 \arrow[ru] & 35 \end{tikzcd}$$

2nd. To find the reciprocal, ***invert*** **the fraction** –
Change the places of the numerator and the denominator

*** The Test of Reciprocal:** *"Two numbers are reciprocals of each other, if their product is 1."*

$$\frac{\cancel{5}}{\cancel{1}} \times \frac{\cancel{1}}{\cancel{5}} = 1 \quad \text{and} \quad \frac{\cancel{35}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{35}} = 1$$

The factors cancel out.

Connection: You need this knowledge in doing **division of fractions.**

Reciprocal or Multiplicative Inverse: Changing Division To Multiplication

Multiplicative inverse is another name for reciprocals. The following example shows you how the reciprocal or the multiplicative inverse is used to change division to multiplication.

Example: Divide 6 ÷ 3/4

$$6 \div \frac{3}{4} = \overset{\textcircled{1}}{\frac{6}{\frac{3}{4}}} = \overset{\textcircled{2}}{\frac{6 \times \frac{4}{3}}{\cancel{\frac{3}{4}} \times \cancel{\frac{4}{3}}}} = \overset{\textcircled{3}}{\frac{6 \times \frac{4}{3}}{1}} = \overset{\textcircled{4}}{6 \times \frac{4}{3}}$$

① Write the division as **a complex fraction**, (See p.91)

② ***Multiply both*** the numerator and the denominator of the complex fraction **by 4/3, the reciprocal of the denominator 3/4.**

③ The denominator **equals 1** (The product of 3/4 and its reciprocal 4/3 equals 1)

④ ***Therefore, To divide by a number is the same as to multiply by its reciprocal.***

Connection: In Algebra, we do not use division. We change division to multiplication **by using the property of multiplicative inverse** as seen above.

Summary
(Factors & Multiples)

* Factors are exact divisors. The greatest common factor (GCF) is the greatest factor that is common to both the numerator and the denominator of a given fraction.
* Multiples are the products of a number and any natural numbers. The least common multiple (LCM) of two or more numbers is the least multiple that is common to those numbers.
* We use division to find the factors of a given number Prime factorization is the process of writing a composite number as the product of prime factors.
* The "1 property of multiplication & division" with 1 written in fraction form allow us to write equivalent fractions.
* Cancellation is used to simplify computation by eliminating the common factors before multiplying.
* To write equivalent fraction(s) of lower terms, we divide both the numerator and the denominator of a given fraction by their common factor By using GCF, we reduce a fraction to lowest terms.
* To write equivalent fraction(s) of higher terms, we multiply both the numerator and the denominator of a given fraction by the same non-zero number By using LCM, we change unlike fractions to like fractions.
* Prime factorization method can be used to find both the GCF and the LCM.
* To write the reciprocal of a fraction is to invert the numerator and the denominator of the fraction.

Part IV. Fraction Operations

C. Addition

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General Procedure For Adding Fractions

To Add Like Fractions:

1st. Add the numerators. (The sum is the new numerator of the fraction.)

2nd. Place the sum (step 1) over the same common denominator.

3rd. Reduce the fraction to lowest terms. For example:

(a) 1/4 + 1/4 = 2/4 = 1/2	(reduce to its lowest term)
(b) 3/7 + 4/7 = 7/7 = 1	(change improper fraction to 1)
(c) 5/8 + 7/8 = 12/8 = 1 4/8 = 1 1/2	(change to mixed number)

To Add Unlike Fractions:

A. If the larger denominator is a multiple of the smaller one, use the larger denominator as the least common denominator (LCD) of the given fractions.

1st. Write an equivalent fraction for the other fraction using the LCD as the denominator.

2nd. Add the like fractions by following the steps given above.

B. If *no* denominator is a multiple of the other, then,

1st. Find the least common denominator of the given fractions (use LCM). (p.317)

2nd. Write equivalent fractions for the given fractions using the LCM as the common denominator.

3rd. Add the like fractions by following the steps given above.

General Procedure For Adding Mixed Numbers

To Add Mixed Numbers with Like Fractions:

1st. Add the whole number parts of the fractions.
2nd. Add the fractional parts by following the steps given on last page.
3rd. Add the sum of the whole numbers and the sum of the fractions.

To Add Mixed Numbers with Unlike Fractions:

A. If one denominator is a multiple of the other, use that denominator as the least common denominator.
  1st. Write equivalent fraction with the same denominator for the other.
  2nd. Add the like fractions by following the steps given on last page.
  3rd. Add the whole number parts.
  4th. Add the sum of the whole numbers and the sum of the fractions.

B. If ***no*** denominator is a multiple of the other, then,
  1st. Find the least common denominator of the given fractions using the LCM.
  2nd. Write equivalent fractions so all fractions have the same denominator.
  3rd. Add the like fractions by following the steps given on last page.
  4th. Add the whole number parts.
  5th. Add the sum of the whole numbers and the sum of fractions.

Note: The addition and subtraction of fractions can be done either horizontally or vertically; but the multiplication and division of fractions should be done horizontally

Adding Like Fractions

Example: Add 1/8 + 5/8 + 7/8

$$\frac{1}{8} + \frac{5}{8} + \frac{7}{8} = \frac{1 + 5 + 7}{8} = \frac{13}{8} = 1\frac{5}{8}$$

To add like fractions, it is much easier to do it **horizontally** (side way). Adding like fractions is a simple operation.

Procedure:

1st. Add the numerators: 1 + 5 + 7 = 13 ← **the sum**

2nd. Place the sum over the same common denominator: 13/8

3rd. Change the improper fraction to a mixed number: 1 5/8

(A mixed number is a whole number plus a fraction with "+" omitted.)

Review: To change a mixed number to a fraction

$$\begin{array}{rrrl} & & 1 & 5/8 \quad \text{(remainder/divisor)} \\ \text{divisor} \nearrow & 8\,) & \overline{13} & \\ & & \underline{-\ 8} & \\ & & 5 & \leftarrow \text{remainder} \end{array}$$

Adding Mixed Numbers With Like Fractions

Example: Add 2 4/5 + 7 2/5 + 3 1/5

Horizontal Method

$$2\frac{4}{5} + 7\frac{2}{5} + 3\frac{1}{5}$$

$$= 12 + \frac{7}{5} = 12 + 1 + \frac{2}{5} = 13\frac{2}{5}$$

Vertical Method

$$\begin{array}{r} 2\ 4/5 \\ 7\ 2/5 \\ +\ 3\ 1/5 \\ \hline 12\ 7/5 \end{array} = 13\ 2/5$$

To add mixed numbers, **vertical method seems to be a better method.**
We add the whole number parts and **fractional parts separately.**

Procedure:

1st. Add the whole number parts: 2 + 7 + 3 = 12 ← **the sum**

2nd. Add the fractional parts: $\frac{4 + 2 + 1}{5} = \frac{7}{5} = 1\frac{2}{5}$ ← **the sum**

3rd. Add the sum of whole numbers and the sum of fractional parts:

$$12 + \frac{7}{5} = 12 + 1 + \frac{2}{5} = 13\frac{2}{5} \quad (7/5 = 1\ 2/5 = 1 + 2/5)$$

Adding Whole Number And Like Fractions

Example: Add 7 + 1/9 + 5/9

Horizontal Method

$$7 + \frac{1}{9} + \frac{5}{9}$$

$$= 7 + \frac{6}{9} = 7 + \frac{2}{3} = 7\frac{2}{3}$$

Vertical Method

$$\begin{array}{r} 7 \\ 1/9 \\ +\quad 5/9 \\ \hline 7\ 6/9 \end{array} = 7\ 2/3$$

Procedure:

1st. Add the like fractions: $\frac{1 + 5}{9} = \frac{6}{9}$

2nd. Reduce the fraction, 6/9, to lowest terms:

$$\frac{6}{9} = \frac{6 \div 3}{9 \div 3} = \frac{2}{3}$$

Divide the numerator and the denominator by their common factor. 3/3 = 1

3rd. Add the whole number and the fraction:

$$7 + \frac{2}{3} = 7\frac{2}{3}$$

Write the whole number and the fraction as *a mixed number.*

Adding Unlike Fractions

Example: Add 3/8 + 5/16 + 3/4

Horizontal Method:

$$\frac{3}{8} + \frac{5}{16} + \frac{3}{4}$$

$$= \frac{6}{16} + \frac{5}{16} + \frac{12}{16} = \frac{23}{16} = 1\frac{7}{16}$$

Vertical Method:

$$\begin{array}{rcccl} \frac{3}{8} & = & \frac{3 \times 2}{8 \times 2} & = & \frac{6}{16} \\ \frac{5}{16} & & & = & \frac{5}{16} \\ + \frac{3}{4} & = & \frac{3 \times 4}{4 \times 4} & = & \frac{12}{16} \\ \hline & & & & \frac{23}{16} = 1\frac{7}{16} \end{array}$$

Procedure:

1st. **Find the LCM of 8, 16, and 4.** The LCM (8, 16, 4) is 16
We know: *16 is a multiple of 8* (8 x 2 = 16) and
16 is also a multiple of 4 (4 x 4 = 16).

2nd. **Write equivalent fractions for 3/8 and 1/4** ***with 16 as the denominator.***
Multiply 3/8 **by 2/2** and 3/4 **by 4/4.** (See Vertical Method)

3rd. Add the like fractions.

4th. Change the **improper fraction** to **a mixed number.**

Adding Mixed Numbers With Unlike Fractions

Example: Add 4 2/3 + 6 1/3 + 2/9

Horizontal Method:

$$4\frac{2}{3} + 6\frac{1}{3} + \frac{2}{9}$$

$$= 4\frac{6}{9} + 6\frac{3}{9} + \frac{2}{9} = 10\frac{11}{9} = 11\frac{2}{9}$$

Vertical Method:

$$\begin{array}{rcr} 4\frac{2}{3} & = & 4\frac{6}{9} \\ 6\frac{1}{3} & = & 6\frac{3}{9} \\ +\ \frac{2}{9} & = & \frac{2}{9} \\ \hline & & 10\frac{11}{9} = 11\frac{2}{9} \end{array}$$

Procedure:

1st. Find the LCM of 3 and 9. The LCM of 3 and 9 is 9.
We know ***9 is a multiple of 3.***

2nd. Write equivalent fractions for 2/3 and 1/3 *with 9 as the denominator.*
Multiply 2/3 and 1/3 by 3/3 respectively.

3rd. Add the like fractions. The sum is 11/9.

4th. Change the improper fraction, 11/9, to a mixed number: 1 2/9

5th. Add the whole numbers and the fraction: 10 + 1 + 2/9 = 11 2/9

Part IV. Fraction Operations

D. Subtraction

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General Procedure For Subtracting Fractions (Review first pp.313-318)

Keep in mind that **fractions must have the same common denominators,** before they can be added, subtracted, or even compared. **Make sure you know how to find LCM/LCD before you do the subtraction of the unlike fractions.**

Subtract Like Fractions:

1st. Subtract the numerators - (**The difference** is the new numerator.)
2nd. Write the difference of the numerators over the same denominator.
3rd. Always write the answer **in lowest terms.**

Subtract Unlike Fractions:

1st. ***Find the least common denominator (LCD) of the given fractions*.**
Review pages 320-325 before you add or subtract unlike fractions.
2nd. Rewrite the given fractions as like fractions *with the LCD as the common denominator.*
3rd. Subtract the like fractions by following the steps given above.

Subtract Mixed Numbers Without Borrowing:

1st. Subtract the fractional parts.
  a) **If like fractions,** follow the subtraction of like fractions.
  b) **If unlike fractions,** follow the subtraction of unlike fractions.
2nd. Subtract the whole number parts.

General Procedure For Subtracting Mixed Numbers With Borrowing

As with whole numbers and decimals, the subtraction of mixed numbers sometimes involves borrowing. It occurs **when the fraction of the minuend** (the top number) ***is less than*** **the fraction of the subtrahend** (the bottom number). In borrowing, follow the steps given below:

1st. If **unlike fractions**, change them to **like fractions using the LCM.**
2nd. After Step 1, **if the fraction of the bottom number *is larger than* the fraction of the top number, make the fraction of the top number larger** in the following way:

- a) ***Borrow 1 from the whole number.*** (Remember to reduce the whole number by 1.)
- b) **Write the 1 as a like fraction using the same denominator.**
  Example: If the denominator of like fractions is 5, write **1 as 5/5.**
  If the denominator of like fractions is 9, write **1 as 9/9.**
- c) **Add the 1,** written in fraction, **to the fractional part** of the number.
  It makes the minuend an improper fraction.

3rd. Subtract the **fractional parts.**
4th. Subtract the **whole number part.**

Note: we can add or subtract fractions either horizontally or vertically.

Subtracting Like Fractions (Fractions With Like Denominators)

Example: Subtract 8/9 - 2/9

$$\frac{8}{9} - \frac{2}{9} = \frac{8-2}{9} = \frac{6}{9} = \frac{2}{3}$$ Divide 6/9 by 3/3

Procedure:
1st. **Subtract the numerators:** 8 - 2 = 6 **← the difference**
2nd. **Write the difference over the same common denominator:** 6/9
3rd. **Reduce** the fraction **to lowest terms:** 6/9 = 2/3.

Example: 6/7 - 1/7

$$\frac{6}{7} - \frac{1}{7} = \frac{6-1}{7} = \frac{5}{7}$$

Procedure:
1st. Subtract the numerators: 6 - 1 = 5 **← the difference**
2nd. Write the difference over the same common denominator: 5/7

Note: Subtraction of like fractions is as simple as the addition of like fractions.
It is much easier to do it in horizontal way

Subtracting Mixed Numbers With Unlike Fractions

Example: Subtract 6 3/4 - 2 5/7

Remember: Fractions must have **the same denominators** before they can be subtracted (or added, or compared).

$$\begin{array}{rcrcl}
6\frac{3}{4} & = & 6\frac{21}{28} & \longleftarrow & \frac{3}{4} \times \frac{7}{7} = \frac{21}{28} \\
-\;2\frac{5}{7} & & 2\frac{20}{28} & \longleftarrow & \frac{5}{7} \times \frac{4}{4} = \frac{20}{28} \\
\hline
 & & 4\frac{1}{28} & &
\end{array}$$

Procedure:

1st. Change the **unlike fractions to like fractions using the LCM**
Since 4 and 7 are relatively prime (See p.293),
the LCM of 4 and 7 is 28, the product of 4 and 7.

2nd. Write equivalent fractions ***with 28 as the common denominator.***

3rd. Subtract the fractional part.

4th. Subtract the whole number part.

Subtracting A Mixed Numbers From A Whole Number (With Borrowing)

Example: Subtract 5 - 2 3/8

Remember: 1 can be written as a fraction with the numerator equals the denominator.

$$\begin{array}{rcrl} 5 & = & 4\,\frac{8}{8} & \longleftarrow \\ -\;2\,\frac{3}{8} & & -\;2\,\frac{3}{8} & \\ \hline & & 2\,\frac{5}{8} & \end{array}$$

$$5 = 4 + 1 = 4 + \frac{8}{8}$$

(8/8 = 1 because any number divided by itself is 1.)

Procedure:

1st. ***Borrow 1 from 5.*** The whole number 5 becomes 4.
Write the 1 borrowed as a fraction *with 8 as the denominator:*
1 = 8/8 ← an improper fraction

2nd. Subtract the like fractions: 8/8 - 3/8 = 5/8

3rd. Subtract the whole number: 4 - 2 = 2

Note: Subtracting a fraction from a whole number follows the same procedure.

Subtracting Mixed Numbers (With Borrowing)

Example: `Subtract 3 1/4 - 1 3/4`

$$\begin{array}{rcrcl} 3\frac{1}{4} & = & 2\frac{5}{4} & \longleftarrow & 3\frac{1}{4} = (2+1)+\frac{1}{4} = 2+\frac{4}{4}+\frac{1}{4} = 2\frac{5}{4} \\ -\,1\frac{3}{4} & & 1\frac{3}{4} & & \\ \hline & & 1\frac{2}{4} & = & 1\frac{1}{2} \end{array}$$

Procedure:

1st. Since the fraction of the minuend (the top number) *is less than* the fraction of the subtrahend (the bottom number),

- a) ***Borrow 1 from 3,*** the whole number 3 becomes 2.
- b) Write the 1 borrowed as a fraction *with 4 as the denominator:* 1 = 4/4
- c) Add 4/4 to 1/4 to makes it **an improper fraction:** 5/4

2nd. Subtract the fractional part: 5/4 - 3/4 = 2/4

3rd. Subtract the whole number part: 2 - 1 = 1

4th. Reduce the fraction to lowest terms: 2/4 to 1/2

Summary

(Addition & Subtraction)

* Unlike fractions must be changed to like fractions with the same denominators before they can be added or subtracted.
* The order in which we add fractions does not affect the sum, but the order in which we subtract fractions does affect the difference. We always subtract smaller number from a larger number because subtraction is not commutative.
* To add like fractions, add the numerators and write the sum over the common denominator
* To subtract like fractions, subtract the smaller numerator from the larger numerator and write the difference over the same denominator
* To add or to subtract mixed numbers, add or subtract the whole number parts and fractional parts separately. Then add the two sums or differences together
* If subtraction requires borrowing, borrow 1 from the whole number part and reduce that number by 1 Write the 1 borrowed as a like fraction of the fraction you are adding to. This process makes the minuend an improper fraction which is larger than the subtrahend.
* At the end of computation, an improper fraction must be changed to a mixed numbers and the fraction must be written in lowest terms.

Part IV. Fraction Operations

E. Multiplication

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General Procedure For Multiplying Fractions

The sale signs, "1/2 off" and "1/3 off", we see in stores are the examples of multiplication of fractions. **Multiplication of fractions is rather simple.**

Before multiplying, change mixed numbers to improper fractions; whole numbers to fractions. Then follow either method A or method B:

Method A. Multiply First, Then Reduce

Step 1. Multiply the numerators, then multiply the denominators. **Write the product of the numerators over the product of the denominators.**

Step 2. Simplify the fraction or reduce the fraction to lowest terms.

Method B. Cancel First, Then Multiply - A shorter method

Step 1. *Cancel the common factors* before multiplying.

- Look for number(s) that can *divide evenly* into both numerator and denominator. Or
- Use **prime factorization** and cross out the common factors.

Step 2. Multiply the numbers remaining in the numerators. Multiply the numbers remaining in the denominators. Write the product of numerators over the product of the denominators.

Mixed Numbers vs. Multiplication Of Fractions

In working with fractions, some students got mixed up in the following operations. It is important to avoid the similar mistakes.

1. Do not mixed up mixed number with the multiplication of fractions. Example:

(a) $4\frac{5}{8}$ -- (a mixed number) -- $= 4 + \frac{5}{8}$ (with "+" sign omitted)

(b) $4 \text{ x } \frac{5}{8}$ -- (a multiplication) -- $= \frac{4 \text{ x } 5}{8} = \frac{5}{2} = 2\frac{1}{2}$

Note: 4 x 5/8 can be written as 4(5/8), but never as 4 5/8.

2. Cancellation is used only with multiplication of fractions and not with addition. Example:

(a) $\dfrac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}} \text{ x } \dfrac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{9}}}$

To cancel means **to divide** the numerator and the denominator **by their common factor** because division and multiplication are inverse operations.

(b) $\dfrac{3}{4} + \dfrac{8}{9}$

Division and addition ***are NOT*** inverse operations.

Multiplying Whole Numbers By Proper Fractions And Vice Versa

Example: Multiply 2/15 x 3.

Method 1.

$$\frac{2}{15} \times 3 = \frac{2 \times 3}{15} = \frac{6}{15} = \frac{2}{5} \quad \text{Or} \quad \frac{2}{\cancel{15}_{5}} \times \overset{1}{\cancel{3}} = \frac{2}{5}$$

1st. **Multiply the numerator by the whole number.** Write the product over the denominator: 6/15.

2nd. Reduce the fraction to lowest terms.

Or **1st.** **Cancel the common factor:** divide 15 & 3 by 3.

2nd. Write the result as a fraction: 2/5 .

Method 2.

$$\frac{2}{15} \times 3 = \frac{2}{15 \div 3} = \frac{2}{5}$$

1st. **Divide the denominator by the whole number.**

(It's the same as cancel the common factor.)

2nd. Write the result as a fraction: 2/5.

Remember: When one factor is a whole number, you can either multiply the numerator by the whole number or divide the denominator by the whole number, **if the denominator is a multiple of the whole number.**

Multiplying Proper Fractions By Proper Fractions

Example: `Multiply.  15/18 x 3/5`

Method 1.

$$\frac{15}{18} \times \frac{3}{5} = \frac{15 \times 3}{18 \times 5} = \frac{45}{90} = \frac{1}{2}$$

$$\boxed{\begin{array}{r|rr} 5 & 45 & 90 \\ \hline 9 & 9 & 18 \\ \hline & 1 & 2 \end{array}}$$

Multiply the numerators; multiply the denominator.
Write the results as a fraction: 45/90.
Reduce the fraction to lowest terms. (See inside the box.)

Method 2.

$$\frac{\overset{3}{\cancel{15}}}{\underset{6}{\cancel{18}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{5}}} = \frac{3 \times 1}{6 \times 1} = \frac{1}{2}$$

Cancellation using slash.

Slash 15 and 5, divide each number by 5.
Slash 18 and 3, divide each number by 3.
Again divide 3 and 6 by 3, reduce to lowest terms.

Method 3.

$$\frac{15}{18} \times \frac{3}{5} = \frac{(\cancel{3} \times \cancel{5}) \times \cancel{3}}{(\cancel{3} \times \cancel{3} \times 2) \times \cancel{5}} = \frac{1}{2}$$

Cancellation using prime factoring.

Prime factor each number, **then cancel** the common factors.

Multiplying Mixed Numbers By Mixed Numbers

Example: Multiply 3 3/4 x 2 1/2

$$3\frac{3}{4} \times 2\frac{1}{2} = \frac{(3 \times 4) + 3}{4} \times \frac{(2 \times 2) + 1}{2} = \frac{15}{4} \times \frac{5}{2}$$

1st. Change mixed numbers to improper fractions (See p.95)

$$\frac{15}{4} \times \frac{5}{2} = \frac{75}{8} = 9\frac{3}{8}$$

$$\boxed{75 \div 8 = 9\frac{3}{8}}$$

2nd. Multiply the numerators. Multiply the denominator. write the result as a fraction: 75/8
(We *can not* use cancellation because there is no common factor.)

3rd. Change the improper fraction to a mixed number. Divide the numerator by the denominator and write the remainder as a fraction: dividend over divisor.

Multiply Mixed Numbers By Proper Fractions

Change the mixed number to improper fraction first. Then follow the same procedure.

Multiplying Whole Numbers By Mixed Numbers And Vice Versa

Example: Multiply 5 x 6 1/5

$$5 \times 6\frac{1}{5} = 5 \times \frac{(6 \times 5) + 1}{5} = \cancel{5} \times \frac{31}{\cancel{5}} = 31$$

Method 1. Use Improper Fraction

1st. Change the mixed number to an improper fraction (See p 95)

2nd. Cancel the common factor. 5 & 5 cancel out.

$$5 \times 6\frac{1}{5} = 5 \times (6 + \frac{1}{5}) = (5 \times 6) + (\cancel{5} \times \frac{1}{\cancel{5}}) = 30 + 1 = 31$$

Method 2. Use Distributive Property Over Addition

1st. Write the mixed number as the sum of a whole number and a fraction.

2nd. Multiply each addend (6 & 1/5) by 5 respectively.

3rd. Add the products obtained in step 2.

Note: The upper graders should be familiar with the second method. The distributive property is very useful.

The Word "Of" In Fractions Problems (See also "Percents" p.393)

In fraction (also percent) problems, the word **"of" means "times"**.

Example: 1/2 **of** 4/5 means 1/2 **x** 4/5 **"of" means multiplication.**

In general, there are **three basic types of fraction problems.** The following shows how to translate each into mathematic sentence:

Type 1.	**2/3 of 4/7 is what number?**	**To find the answer:**
	2/3 x 4/7 = ?	Simply multiply 2/3 by 4/7.
Type 2.	**3/5 of what number is 2 1/3?**	
	3/5 x ? = 2 1/3	Divide 2 1/3 by 3/5.
Type 3.	**What number of 2 3/8 is 2/5?**	
	? x 2 3/8 = 2/5	Divide 2/5 by 2 3/8.

Remember: Multiplication and division are inverse operations (See p.193)

The Word "Of" In Measurements

The word "of" is used very often in measurement involving fractions. The following units of the measurements are often omitted in word problems, because they are common knowledge and you are expected to know.

* A quarter of a day = $\frac{1}{4}$ x 24 hr = 6 hr (a day = 24 hours)

* A half of a pound = $\frac{1}{2}$ x 16 oz = 8 oz (1 pound = 16 ounces)

* Two third of a yard = $\frac{2}{3}$ x 36 in = 24 in (1 yard = 36 inches)

* A quarter of an hour = $\frac{1}{4}$ x 60 min = 15 min (1 hour = 60 minutes)

* One quarter of a dollar = $\frac{1}{4}$ x 100¢ = 25¢ (1 dollar = 100 cents)

* Three fourth of a year = $\frac{3}{4}$ x 12 = 9 months (1 year = 12 months)

* A quarter of a gallon = $\frac{1}{4}$ x 4 gts = 1 gt (1 gallon = 4 quarts)

* One third dozen = $\frac{1}{3}$ x 12 = 4 (1 dozen = 12)

Think! Can we write a dollar and thirty cents as 130 cents? Can we write 1 hour 30 minutes as 130 minutes? If yes, why yes? If not, why not?

Summary
(Multiplication)

- The multiplication of fractions is the simplest computation of the four operations of fractions.
- In adding and subtracting fractions, the mixed numbers are not changed to improper fractions. But in multiplication, mixed numbers must be changed to improper fractions before they can be multiplied.
- To multiply fractions, multiply the numerators, then multiply the denominators. Write the product of the numerators over the product of the denominators. Then, reduce the fraction to lowest terms or change to a mixed number
- A simpler way of multiplying fractions is to cancel first the factors which are common to both the numerator and the denominator, and then multiply the simplified fractions.
- The word "of" means "time" or "multiply" when it is used in word problems or measurement involving fractions. And the word "is" means "equal to."

Part IV. Fraction Operations

F. Division

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General Procedure For Dividing Fractions

Multiplication of fraction is one of the simplest of the four arithmetic operations. The division of fractions is as simple as the multiplication of fractions.

Step 1. Change any mixed number to an improper fraction.
Change any whole number to a fraction.
(Write the whole number as a fraction is helpful, when you need to invert the whole number divisor.)

Step 2. ***Invert*** **the divisor,** the number **comes after** the division sign.
(To divide by a number is the same as to multiply by its reciprocal. A reciprocal is an inverted number as demonstrated on the next page.)

Step 3. Multiply the fractions by following the steps given on page 352

Remember:
- ***The order*** in which we **multiply** or **add** fractions ***does not*** affect the product or the sum because **both operations are commutative.**
- ***The order*** in which we **divide** or **subtract** fractions ***does*** affect the quotient or the difference because **neither is commutative.**

To divide by a number is the same as to multiply by its reciprocal. - a reciprocal is an inverted number. Here is the reason why:

Example: Divide 5 ÷ 3/4

ⓐ ⓑ ⓒ ⓓ

$$\frac{5}{\frac{\cancel{3}}{\cancel{4}}} \times \frac{\frac{4}{3}}{\frac{\cancel{4}}{\cancel{3}}} = \frac{5 \times \frac{4}{3}}{1} = 5 \times \frac{4}{3}$$

ⓐ Write the division as a **complex fraction** (p.91)

ⓑ Multiply **both** numerator and denominator by 4/3, the reciprocal of 3/4, so (4/3)/(4/3) = 1

ⓒ **The denominator becomes 1.**

ⓓ Thus, **5 ÷ 3/4 = 5 x 4/3**

Example: Divide 7/9 ÷ 2/5

$$\frac{7}{9} \div \frac{2}{5} = n$$

Let's use n to stand for the **unknown quotient.**

$$\frac{7}{9} = n \times \frac{2}{5}$$

Division and multiplication are **inverse operation.** (p.193)

$$\frac{7}{9} \times \frac{5}{2} = n \times \frac{\cancel{2}}{\cancel{5}} \times \frac{\cancel{5}}{\cancel{2}}$$

Multiply **each side** of the equal sign by 5/2, **the reciprocal of 2/5.** (p.326)

$$\frac{7}{9} \times \frac{5}{2} = n$$

Thus, we change the division of fraction to a multiplication.

Dividing A Fraction By A Whole Number

Example: Divide 4/5 by 2.

$$\frac{4}{5} \div \frac{2}{1} = ?$$

Write the whole number in fraction form with 1 as the denominator.

$$\frac{\overset{2}{\cancel{4}}}{5} \times \frac{1}{\underset{1}{\cancel{2}}} = \frac{2}{5}$$

Invert the divisor 2/1 and multiply. **Cancel factors before multiplying.**

Dividing A Whole Number By A Fraction

Example: Divide 2 by 4/5.

$$\frac{2}{1} \div \frac{4}{5} = ?$$

It is understood that 2 = 2/1.

$$\frac{\overset{1}{\cancel{2}}}{1} \times \frac{5}{\underset{2}{\cancel{4}}} = \frac{5}{2} = 2\,\frac{1}{2}$$

Invert the divisor 4/5 and multiply. **Cancel factors before multiplying.**

Note: When you ***change the order*** of the numbers, you get the different answers because **division is not commutative.**

Dividing A Fraction By A Fraction

Example: Divide 3/7 ÷ 6/9

$$\frac{3}{7} \div \frac{6}{9} = n$$

Let's *n* stand for the unknown quotient.

$$\frac{\overset{1}{\cancel{3}}}{7} \times \frac{9}{\underset{2}{\cancel{6}}} = \frac{9}{14}$$

Invert the divisor 6/9 and multiply.
Cancel the factors before multiplying.

Dividing A Fraction By A Mixed Number

Example: Divide 7/9 ÷ 2 1/3

$$\frac{7}{9} \div \frac{7}{3}$$

The mixed number must be changed to improper fraction first.

$$\frac{\overset{1}{\cancel{7}}}{\underset{3}{\cancel{9}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{7}}} = \frac{1}{3}$$

Invert the divisor 7/3 and multiply.
Cancel the factors before multiplying.

Remember: All mixed numbers must be changed to improper fractions before multiplying or dividing.

Summary
(Division)

- The order in which we multiply fractions does not affect the product. But the order in which we divide fractions does affect the quotient because division is not commutative.
- Division of fractions, like that of multiplication, requires that mixed numbers must be first changed to improper fractions.
- To divide fractions, invert the divisor then multiply The divisor is the number that comes after the division sign. To invert means to turn the fractions upside down. The denominators become the numerators and the numerators become the denominators.
- To avoid error, write the whole number divisor as a fraction with the number as the numerator and 1 as the denominator Then invert the fraction.

the *family* MATH. companion

ARITHMETIC—THE FOUNDATION OF MATH

Ruth C. Sun

Part V. Ratios, Proportions, Percents

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Ratios, Equal Ratios (see also "Equivalent Fractions" p.300)

A ratio is a **comparison of two numbers or quantities.** It implies a comparison by division. The following is the ratio of 8 boys to 2 girls.

a) 8 **to** 2

b) 8 : 2

c) $\frac{8}{2}$ 8 ← **1st term**, 2 ← **2nd term**

- * A ratio has **two terms: "1st term to 2nd term"**
- * A ratio can be expressed in **three different ways** as shown on the left: **in word**, **in colon**, and **in fraction**. (Fraction form is commonly used.)
- * All three forms are read **"8 to 2."**

Order is important in writing a ratio. **The quantity named first in statement is always the first term.** For example, there are 8 boys and 2 girls:

What is the ratio of **boys** to girls? **8** to 2 or **8**/2 (boys first)

What is the ratio of **girls** to boys? **2** to 8 or **2**/8 (girls first)

Equal Ratios - Ratios like fractions are usually written in lowest terms.

$$\begin{matrix}8\\2\end{matrix} \overset{-\,2}{\underset{-\,2}{=}} \begin{matrix}4\\1\end{matrix} \qquad 8 : 2 = 4 : 1$$

We simplify ratios, write equal ratios,and change ratios to a common denominator like we do with fractions. But we can not change a ratio such as 4/1 or 5/2 to a whole number or a mixed number.

Ratios & Rates, Unit Rate & Unit Price

A rate is a ratio **comparing two quantities in different kinds of units.** You know the comparison is a rate when it uses words like **"per" "for" "each."** The following compares the differences between ratios and rates.

Ratio $\frac{3 \text{ points}}{4 \text{ points}}$ —score of Sue / —score of Tom

"3 **to** 4"
"3 : 4"

In a ratio, the terms are of the same kind or in the same unit .

Rate $\frac{60 \text{ miles}}{1 \text{ hour}}$ — distance / — time

60 **miles per (1)hr**
60 **miles** : 1 **hr**

In a rate, the terms are always in different units.

In ratios, same units cancel out. But in rates, different units are written as part of the rate.

Unit Rate & Unit Price - In unit rate or unit price **the second term is 1 unit.** Unit Rate is the rate that **compares quantity to 1 unit.** Unit price is the rate that compares price to 1 unit.

Rate:	"120 pages in 3 hours"
Unit Rate:	"40 pages/1 hr"
Price:	"$9 for 2 lbs"
Unit Price:	"$4.50/1 lb"

To find the unit rate or unit price, divide the number in the first term by the number in the second term:
120 ÷ 3 = 40; 9 -- 2 = 4.5

Ratios & Proportions

A ratio is a comparison of two numbers. A proportion is **an equation** showing **that two ratios are equal** and it can be written in the following ways:

a) $48 : 12 = 4 : 1$

Both read as "48 is to 12 as 4 is to 1"

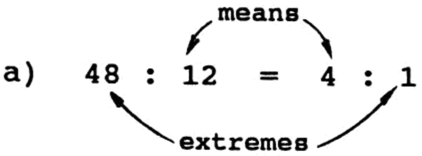
b) $\frac{48}{12} = \frac{4}{1}$ ← **The 1 can not be dropped in a ratio.**

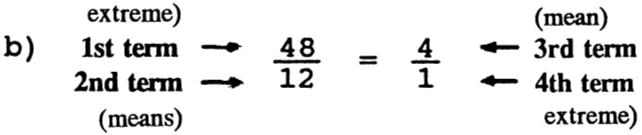
A ratio has two terms but a proportion has 4 terms. We call

- the **1st** and **4th terms** (last term) the **extremes** of the proportion,
- the **2nd** and **3rd terms** (two middle terms) the **means** of the proportion.

In a true proportion: **The product of the means equals the product of the extremes.**

(2nd term) x (3rd term) = (1st term) x (4th term)





We can determine whether two ratios are equal by following methods:

1. **Reduce each ratio to lowest terms and compare.** **(See p309)**
2. **Use the cross products.** **(See next page)**

Cross Products - A Test of Equal Ratios or Equivalent Fractions (see also p.303)

We can use cross products (also called cross multiplication) to determine whether two ratios or two fractions are equal. We know in a true proportion, **the product of the extremes equals the product of the means.**

$$\overset{①}{\frac{20}{50}} \overset{?}{\times} \overset{②}{\frac{18}{45}}$$

① 20 x 45 = **900** ← **The products of the extremes**

② 50 x 18 = **900** ← **The products of the means**

$$\frac{20}{50} = \frac{18}{45}$$

Since the cross products are equal, the two ratios are equal. Thus, 25/50 = 18/45 is a proportion.

We can use proportion and cross products to **find the value of the "unknown"** term if we know the other three terms. Follow the steps given below:

① $\frac{14}{21} = \frac{6}{n}$

② $14 \times n = 21 \times 6$

③ $n = \frac{21 \times 6}{14} = 9$

④ $\frac{14}{21} = \frac{2}{3}, \quad \frac{6}{9} = \frac{2}{3}$ (=)

① **Set up a proportion:** with an unknown term "n" in one of the ratio.
② **Cross multiply:** the products of the extremes = the products of the means.
③ **Solve for n:** divide the product of the means by the given extreme.
④ **Check:** reduce each ratio to lowest terms and compare. Check (√).

Setting Up Proportions In Different Ways (Review first p.374)

In the following, we use the proportion **4 : 2 = 14 : 7** as an example to show that there are **more than one way** of setting up a proportion. We can use the following formula to test whether each proportion is a true proportion:

2nd term x 3rd term = 1st term x 4th term

(a)	1st term 2nd term	$\frac{\textcircled{4}}{\boxed{2}} \overset{?}{=} \frac{\boxed{14}}{\textcircled{7}}$	3rd term → 4th term →	2 x 14 = 28 4 x 7 = 28	true proportion
(b)	2nd term 1st term	$\frac{\boxed{2}}{\textcircled{4}} \overset{?}{=} \frac{\textcircled{7}}{\boxed{14}}$	4th term → 3rd term →	4 x 7 = 28 2 x 14 = 28	true proportion
(c)	2nd term 4th term	$\frac{\boxed{2}}{\textcircled{7}} \overset{?}{=} \frac{\textcircled{4}}{\boxed{14}}$	1st term → 3rd term →	7 x 4 = 28 2 x 14 = 28	true proportion
(d)	4th term 2nd term	$\frac{\textcircled{7}}{\boxed{2}} \overset{?}{=} \frac{\boxed{14}}{\textcircled{4}}$	3rd term → 1st term →	2 x 14 = 28 7 x 4 = 28	true proportion

Scale Drawings (Review "Proportions" "Cross Products" p.374)

A scale drawing reduces the size of an actual object proportionately so that the object appears in the same shape but in a much smaller size. Scale drawings are used to make maps, draw blue prints for buildings, cars, plane, etc. **It utilizes the concepts of proportions and cross multiplication.**

Here is how to use proportion to find the actual distance of the city 13 inches away on the map, if the scale is 1 inch for 20 miles.

$$\begin{array}{rcccl} \text{distance in drawing (inch)} \text{ —— } & 1 & & 13 & \text{ —— distance in drawing (inch)} \\ & \overline{} & = & \overline{} & \\ \text{actual distance (mile)} \text{ —— } & 20 & & n & \text{ —— actual distance (mile)} \end{array}$$

The units of the second ratio must correspond to the units of the first ratio. (See p.373)

cross multiply $1 \times n = 13 \times 20$ $n = 260$ miles

check $1 \times 260 = 260$; $13 \times 20 = 260$ or $13/260 = 1/20$

Step 1. Set up a proportion. Use **the given scale** to write the **first ratio:** 1/20 and use n to represent the unknown distance.

Step 2. Find cross products. Solve for n.

Step 3. Check. Substitute 260 for n and multiply or simplify. Check (√)

The answer: The city is 260 miles away.

Meanings of Percents and The Percent Symbol (Sign) (%)

The word percent comes from **"per"** and **"centum." "Centum" means 100.** So, percent means "by the hundredths" (the number of hundredths of a number), "per 100," "out of 100," "on the basis of 100," "compared with 100," or "divided by 100." The symbol for percent is %. Study the following carefully and find what the percent symbol % stands for:

$$5\% = \frac{5}{100} = 5 \div 100 = 5 \times \frac{1}{100},$$

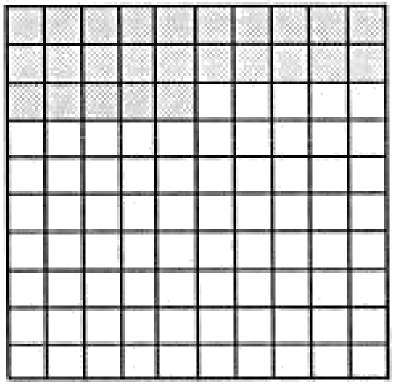
$$1\ \% = \frac{1}{100} = 1 \div 100 = 0.01$$

$$100\% = \frac{100}{100} = 100 \div 100 = 1$$

Points To Remember: **(See "Shortcut..." p.390)**

- **To drop the % sign means to divide the number by 100** or to multiply by its reciprocal 1/100 (= 0.01). **(See "Reciprocal" p.326)**
- **To add the % sign means to divide the number by 100.** Therefore, you first multiply the number by 100 before you add the % (= to multiply by 100/100).
- **Any number with the % sign should be changed** to a decimal, or a fraction, or a ratio before it is used in computation.

Relating Percents to Ratios, Fractions, & Decimals



On the left is a 10-by-10 grid, 25 out of 100 squares are shaded. When we compare the shaded squares to the total of 100 squares, we can express the relation as:

* **a percent:** 25% (25 per 100)
* **a ratio:** 25:100 or 25/100
* **a fraction:** 25/100
* **a decimal:** 0.25

The example shows the basic relationships that exist among percent, ratio, fraction, and decimal. They are different forms of writing the same number. **(See p.98)** We can say:

- Percent is a special kind of **ratio** which always **compares with a number 100.**
  (e.g.) 25% means "25 compared with 100" -- 25:100 or 25/100
- Percent is a special kind of **fraction** whose **denominator is always 100.**
  (e.g.) 25% means "25 out of 100" -- 25/100
- Percent is a special kind of **decimal** with the place value of **"hundredths."**
  (e.g.) 25% means "25 hundredths (of 100)" -- 0.25

Percents and Decimal/Fraction Equivalents (Review also p.98)

Since percents, decimals, and fractions are different forms of writing the same number, you should not only know how to change from one form to another but also remember the equivalent forms of the most commonly used percents like the one listed below.

Percent	Decimal Equivalent	Fraction Equivalent
1%	0.01	1/100
5%	0.05	1/20
10%	0.1	1/10
20%	0.2	1/5
25%	0.25	1/4

Percent	Decimal Equivalent	Fraction Equivalent
40%	0.4	2/5
50%	0.5	1/2
60%	0.6	3/5
80%	0.8	4/5
100%	1	1

Note: To change **a mixed number percent,** $12\frac{1}{2}\%$, to a decimal, first write it as a **sum** of a whole number percent and a fraction percent:

$$12\frac{1}{2}\% = 12\% + 1/2\% = 12\% + 0.5\% = 0.12 + 0.005 = 0.125$$

Fractions and Decimal/Percent Equivalents

Also for your quick reference, here are the decimal/percent equivalents of some of the common fractions which you are likely to encounter in working with numbers.

Common Fraction	Decimal Equivalent	Percent Equivalent
1/2	0.5	50%
1/3	0.33	$33\frac{1}{3}$%
1/4	0.25	25%
1/5	0.2	20%
1/6	0.16	$16\frac{2}{3}$%
1/8	0.125	$12\frac{1}{2}$%

Common Fraction	Decimal Equivalent	Percent Equivalent
2/3	0.66	$66\frac{2}{3}$%
3/4	0.75	75%
5/6	0.83	$83\frac{1}{3}$%
3/8	0.375	$37\frac{1}{2}$%
5/8	0.625	$62\frac{1}{2}$%
7/8	0.375	$37\frac{1}{2}$%

Example: Find $16\frac{2}{3}$% of 240.

1/6 x 240 = 40 $16\frac{2}{3}$% = 1/6 and the word "of" means "times."

Percents, Decimals, & Fractions

Keep in mind that decimals and percents are **two special types** of fractions which differ from the common fractions The differences are found in their denominators:

$\frac{2}{5} \quad \frac{3}{4} \quad \frac{1}{25}$ ←

In **Common Fractions,** the denominators are any non-zero number except the powers of 10.

$0.04 = \frac{4}{10} \quad 0.25 = \frac{25}{100}$ ←

In **Decimals**, the denominators are the powers of 10. Instead of writing the denominators, they are indicated by the position of the digit. **(See p.98)**

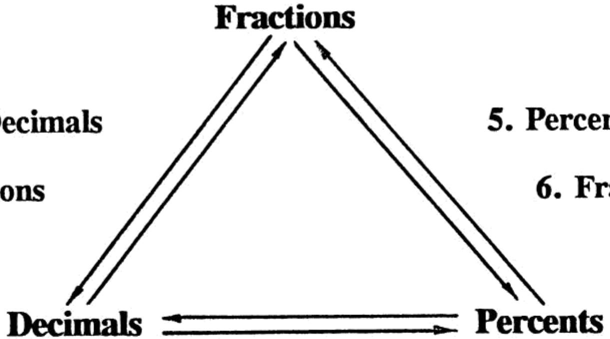
$5\% = \frac{5}{100} \quad 20\% = \frac{20}{100}$ ←

In **Percents,** the denominators are always 100. Instead of writing the denominator, they use a percent sign % (% = 1/100). **(See p.378)**

> **Fractions are always written in the lowest terms. But when a fraction represents a percent, you can not simplify it. For example, you can not write 5/100 & 20/100 above as 1/20 & 1/5.**

Interchanging Among Fractions, Decimals, & Percents

Since fractions, decimals, and percents are **different forms of writing the same number,** it is important that you know how to change from one form to another. The following shows the **six possible interchanges** and "How-To" examples are given on the successive pages.



1. **Fractions —> Decimals**
2. **Decimals —> Fractions**
3. **Decimals —> Percents**
4. **Percents —> Decimals**
5. **Percents —> Fractions**
6. **Fractions —> Percents**

1. Changing Fractions to Decimals (Review first p.98)

How to change a fraction to a decimal depends on the denominator of the fraction as shown below. If you are not sure, **you can always change a fraction to a decimal by dividing the numerator by the denominator.**

$$\frac{2}{10} = 0.2 \qquad \frac{25}{100} = 0.25$$

* **For Decimal Fractions**

Write directly as a decimal the way the fraction is read **(See p.98)**

Clue: The decimal places should equal the number of zeros in the denominator

$$\frac{3}{4} = 3 \div 4 = 0.75$$

* **For Common Fractions**

Division changes any fraction to a decimal. But the result can be:

a) **A terminating decimal** - If the given fraction can be changed to a **equivalent decimal fraction.** Then write as a decimal. **(p.266)**

a) $$\frac{3}{4} \; \overset{\mathrm{x}\,25}{\underset{\mathrm{x}\,25}{=}} \; \frac{75}{100} = 0.75$$

b) **A repeating decimal** - If the given denominator **does not** divide evenly into 10, 100, etc. **(p.267)**

b) $$\frac{1}{3} = 1 \div 3 = 0.33...$$

2. Changing Decimals to Fractions

Remember: The denominator of a decimal is always a power of 10 (p.382)

*** For Decimal Less Than 1 (< 1):**

1st. Write the **decimal as the numerator.** (Drop zero before whole number p.66)

2nd. Write **the place value of the last digit as the denominator.**

3rd. Reduce to lowest terms.

$$0.04 = \frac{4}{100} = \frac{1}{25}$$

hundredths — reduce

*** For Decimal Greater Than 1 (> 1):**

1st. Write the mixed decimal **as the sum of whole number and decimal** (p.224)

2nd. Keep the whole number part. But **change the decimal part to fraction** following the method given above.

3rd. Write as a mixed numbers **with** "+" **omitted.** Simplify the fraction.

$$2.105 = 2 + 0.105$$

$$2 + 0.105 = 2 + \frac{105}{1000}$$

$$2 + \frac{105}{1000} = 2\,\frac{105}{1000} = 2\,\frac{21}{200}$$

Suggestion: Read **"Decimals & Decimal Fractions"** (p 98) & **"Changing Fractions to Decimals"** (p.266) together with last page and this page.

3. Changing Decimals to Percents

Remember: Percent fractions always have 100 as the denominator.

Change 0.5 to a percent.

Method 1. Use "1" Property

1st. Multiply 0.5 by 100/100 (= 1) changing the decimal to a fraction with the denominator of 100.

2nd. Write the fraction as the numerator times 1/100: 50 x 1/100
(A reverse process of multiplying fraction.)

3rd. Replace 1/100 with %. (See p.378)

$$0.5 \times \frac{100}{100} = \frac{50}{100}$$

$$\frac{50}{100} = 50 \times \underbrace{\frac{1}{100}} = \underbrace{50\%}$$

Method 2. Use Equivalent Fraction

1st. Write 0.5 as a decimal fraction.

2nd. Multiply 5/10 by 10/10 (= 1) changing it to a fraction with the denominator of 100.

3rd. Write as a percent. (See above)

$$0.5 = \frac{5}{10} \underset{\times 10}{\overset{\times 10}{=}} \frac{50}{100} = 50\%$$

Shortcut: Move the decimal point two places to the right and add the %. (See p.391)

$$0.5 \longrightarrow 0.\underset{\smile}{5}\,\underset{\smile}{0} \longrightarrow 50\%$$

4. Changing Percents to Decimals

Remember: $\% = (1)\% = 1/100 = 0.01$ (p.378)

$$6\% = 6 \times \frac{1}{100} = \frac{6}{100}$$

$$\frac{6}{100} = 6 \div 100 = 0.06$$

$$6\% = 6 \times 0.01 = 0.06$$

* **For Percents Less than 10%**

Method 1. Change the % to 1/100

Drop the % and multiply by 1/100 changing the percent to a fraction with a denominator of 100. Then divide.

Method 2. Change the % to 0.01

Drop the % and multiply it by 0.01.

$$212\% = \frac{212}{100} = 2\,\frac{12}{100}$$

$$2\,\frac{12}{100} = 2.12$$

$$212\% = 212 \times 0.01 = 2.12$$

* **For Percents Greater than 100%**

Method 1. Change the % to 1/100

First, change the percent to a fraction. Then, write improper fraction as a mixed number, and then as a mixed decimal.

Method 2. Change the % to 0.01

$$6\% \longrightarrow .0\,6 \longrightarrow 0.06$$

Shortcut: Drop the % and move the decimal point two places to the left (p.378)

5. Changing Percents To Fractions

Remember: % = (1)% = 1/100 = 0.01

Change 45% to a fraction.

$$45\% = 45 \times \frac{1}{100} = \frac{45}{100}$$

$$\underbrace{\frac{45}{100}}_{\text{decimal fraction}} = \frac{45 \div 5}{100 \div 5} = \underbrace{\frac{9}{20}}_{\text{common fraction}}$$

Shorter Way: Change to Fraction

1st. **Percent to fraction:** 45% = 45/100
45% means "45 out of 100" or "45 times 1/100".

2nd. **Fraction in lowest terms**
Change the decimal fraction to a common fraction.

Remember: Numbers end in 5 and 0 are divisible by 5 which means they have 5 as a common factor

$$45\% = 45 \times 0.01 = .45$$

$$.45 = \frac{45}{100} = \frac{9}{20}$$

Longer Way: Change to Decimal first

1st. **Percent to decimal:** 45% = .45
Drop the % & move the decimal point two places to the left.
(See p.390)

2nd. **Decimal to decimal fraction:** 45/100

3rd. **Fraction in lowest terms.**

6. Changing Fractions to percents

$$\frac{3}{4} \underset{\times 25}{\overset{\times 25}{=}} \frac{75}{100} = 75\%$$

Method 1. Use Equivalent Fraction

Write an **equivalent fraction** with a denominator of 100.
Write as a percent.**(See #3, p.386)**

Use Method 1 if the given denominator is a factor of 100: 1, 2, 4, 5, 10, 20, 25, 50, 100.

$$\frac{5}{8} = 5 \div 8 = 0.625$$

$$0.625 \times \frac{100}{100} = 62.5\%$$

Method 2. Use Division

Change the fraction to a decimal by division.**(See #1, p.384)**
Multiply the decimal **by 100/100 (= 1).**
Write as a percent.**(See #3, p.386)**

By using Method 2, we can change any type of fraction to a percent.

$$\frac{5}{8} \overset{?}{=} \frac{n}{100} \qquad \begin{array}{l} \longrightarrow\ 8 \times n \\ \longrightarrow\ 5 \times 100 \end{array}$$

$$\begin{aligned} 8 \times n &= 5 \times 100 \\ n &= 500 - 8 = 62.5\% \end{aligned}$$

Method 3. Use Proportion **(See p.374)**

Write **an equation** making 5/8 equal to **n/100 (an unknown percent).**
Find the **cross products** and solve for n **(n equals percent).**

Shortcut For Changing Percents To Decimals (Review first p.378)

Shortcut: "*Drop the %* and **move the decimal point two places** ***to the left.***"

Here is the reason why: **[Remember: % = (1)% = 1/100 = 0.01]**

$$[\% = 1/100] \quad 45\% = 45 \times \overset{(1)}{\frac{1}{100}} = \overset{(2)}{\frac{45}{100}} = \overset{(3)}{45 \div 100} = \overset{(4)}{0.45}$$

(1) To drop the % means to multiply the number by 1/100: 45 x 1/100
(2) To multiply the number by 1/100 change it to a fraction with a denominator of 100: 45/100
(3) **Fractions means division:** 45 ÷ 100
(4) **To divide by 100,** move the decimal point two places ***to the left:***

$$45 \longrightarrow .4\,5 \longrightarrow .45 \text{ (or } 0.45)$$

$$[\% = 0.01] \quad 45\% = 45 \times \mathbf{0.01} = .45 \text{ (or } 0.45)$$

Or, drop the % means to **multiply** the number **by 0.01** as shown above.

As you see, ***"to drop the %"*** means either *to divide* **the number by 100** or *to multiply* **it by 0.01.** In either case, **move the decimal point** ***two places to the left.***

Shortcut For Changing Decimals To Percents (Review first p.228)

Shortcut: **"Move the decimal point two places** *to the right* and *add the %."*

Here is the reason why: **[Remember: Percents always has a denominator of 100.]**

$$0.72 = \overset{(1)}{0.72 \times \frac{100}{100}} = \overset{(2)}{\frac{72}{100}} = \overset{(3)}{72 \times \frac{1}{100}} = \overset{(4)}{72\%}$$

(1) Multiply the number **by 100/100 (= 1):** 0.72 x 100/100
(2) Multiply the number by 100/100 change it to a percent fraction with a denominator of 100: 72/100
(3) Write the fraction as **the numerator time 1/100:** 72 x 1/100
(4) Write **1/100 as the %** and add it to the number: 72%

$$0.72 \longrightarrow 0.72 \times 100 = 72 \longrightarrow 72\%$$

Or, multiply the number by 100 first and then **add the %** (= x 1/100)

Do you see that **"to move the decimal point** ***two places to the right"*** means ***to multiply*** **the number by 100.** And **"to add the %"** means **to multiply it again by 1/100.** It is the same as multiplying by 100/100 (= 1).

Changing Mixed Number Percents to Fraction/Decimal Equivalents

Changing 87 1/2% to A Fraction:

$$87\frac{1}{2}\% = 87\frac{1}{2} \overset{(1)}{\div 100} = \frac{\overset{7}{\cancel{175}}}{2} \times \overset{(2)}{\frac{1}{\underset{4}{\cancel{100}}}} = \overset{(3)}{\frac{7}{8}}$$

87 1/2% can be written as a complex fraction: 87 1/2 over 100.

(1) Drop the % and *divide* the mixed numbers **by 100:** 87 1/2 ÷ 100
(2) ***To divide by 100*** is the same as ***to multiply by its reciprocal 1/100.***
Change the mixed number to an improper fraction and **cancel first.**
(3) 87 1/2 % equals 7/8. **(Check: 7/8 = 7 ÷ 8 = 0.875 = 87 1/2 %)**

Changing 87 1/2% to A Decimal:

$$87\ 1/2\% = \overset{(1)}{87\% + 1/2\%} = \overset{(2)}{87\% + 0.5\%} = \overset{(3)}{.87 + .005} = \overset{(4)}{0.875}$$

(1) Write the mixed number percent as the **sum** of the whole number percent and the fractional percent.
(2) Change the fractional percent to decimal percent. (1/2 = 1 ÷ 2 = 0.5)
(3) Write both percents as decimals. (See "Shortcut" p.390)
(4) 87 1/2% equals 0.875 . **(Check: 0.875 = 875/1000 = 7/8)**

A General Percent Equation

If we use **R** (ratio or percentage), **T** (total or base), and **P** (part) to represent three terms or quantities in a percent problem, a general statement of percents and its equation will be as follows:

General statement: "Certain ratio **of** total **is** part."

$$\begin{array}{ccccc} \text{Certain ratio} & \textbf{of} & \text{total} & \textbf{is} & \text{part.} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{R} & \mathbf{x} & \mathbf{T} & = & \mathbf{P} \end{array}$$

Equation: **R x T = P**

The word "of" means "times" (x).
The word "is" means "equals" =).

The general statement means "Percentage times total equals its actual quantity "

A Proportion for Percent Problems (Review "Proportions" p.374)

One way of solving percent problems is using **equations.** Another is using a **proportion.** A proportion has four terms. In a percent problem one of the terms is always 100 and if two other terms are known, then a proportion can be used to find the fourth term. The following is the proportion used for solving percent problems. **Make sure the terms correspond!**

$$\begin{array}{r}\text{part} —\\ \text{total} —\end{array} \frac{\mathbf{R}}{\mathbf{100}} = \frac{\mathbf{P}}{\mathbf{T}} \begin{array}{l}— \text{part}\\ — \text{total}\end{array}$$

The proportion is read: "R compares with 100 is the same as P compares with T "

Three Basic Types of Percent Problems (Review first the previous page.)

There are three basic types of percent problems because a percent problem is made up of only three quantities. All three types of problems has to do with finding one of those three quantities when the other two are known. In the following ***n*** represents the unknown quantity.

Three Basic Types of Problem	**Sample Questions**
1. **Finding the part (P):** $\boxed{\mathbf{R\ x\ T} = \boldsymbol{n}}$ 60% x 50 = n	* 60% of 50 is what number? * What (number) is 60% of 50? * Find 60% of 50.
2. **Finding the ratio (R):** $\boxed{\boldsymbol{n}\ \mathbf{x\ T} = \mathbf{P}}$ n x 50 = 30	* What percent of 50 is 30? * What percent is 30 of 50? * 30 is what percent of 50?
3. **Finding the total (T):** $\boxed{\mathbf{R\ x}\ \boldsymbol{n} = \mathbf{P}}$ 60% x n = 30	* 60% of what number is 30? * 30 is 60% of what number? * 30 is 60% of a number, find the number.

6. In general, the simplest way to find the part (P) (Type 1 problem) is to change the percent to a decimal or a fraction and multiply. Change to a fraction if cancellation can be used. **(See p.304)**

7. **A percent number (the ratio) should be changed to** a fraction, a decimal, or a ratio before it is used in computation. **(See p.378)**

8. In setting up a **proportion, the terms must correspond.** Each ratio compares the part with the total (or base). In a percent problem, one of the terms (base) is always 100. **(See p.393)**

9. To change a percent to decimal, drop the % and move the decimal point two places to the left or multiply the number by 0.01. **(p.390)** When changing a percent to a fraction, reduce it to lowest terms if necessary. **(See p.388)**
   For example, 50% = 0.50 = 0.5 or 50% = 50/100 = 1/2

10. To change a number (whole number or decimal) to a percent, multiply the number by 100 and add a percent sign (%).
    For example: 7 = 7 x 100 = 700%, & 0.24 = 0.24 x 100 = 24%

11. Utilize the **concepts of cancellation** **(See p.304)** and **reciprocal** **(See p.326)** whenever possible, it means less calculation.

Solving Type 1 Problems: Finding the Part (Read first p.394)

Problem: During the clearance sale, all the shoes at Z store are 25% off the regular prices. If the regular price of the pair of shoes which Dick bought is $84, how much did he save?

Analysis: The problem -- "25% (sale) of $84 (base price) is how much?"
The equation -- $25/100 \times 84 = n$ and $n = 25/100 \times 84$
The question asked -- "How much did Dick save?"

Clue: If you remember that **25% = 1/4** (25 is a quarter of 100) and if you understand the concept of cancellation (see p.304), the problem can be solved with little calculation. See Method 1.

Method 1. Using Equation - Changing the percent to a fraction

$n = \frac{1}{\underset{1}{\cancel{4}}} \times \$\overset{21}{\cancel{84}}$ Change 25% to 1/4 and multiply. Use cancellation.

$n = \$21$ Ans. Dick saved $21.

To know how much Dick paid for the pair of shoes, subtract $21 from $84 (84 - 21 = 63). Dick paid $63 for the shoes.

Method 2. Using Equation - Changing the percent to a decimal

$n = 0.25 \times \$84$ Change 25% to 0.25 and calculate.

$= \$21$ Ans. Dick saved $21.

Method 3. Using Proportion - Changing the percent to a ratio

$\frac{1}{4} = \frac{n}{84}$ **1st.** Set up a proportion. Change 25% to 25/100 = 1/4 (lowest term).

$4 \times n = 1 \times 84$ **2nd.** Find the cross products. Have n on the left of the equation sign.

$\frac{4 \times n}{4} = \frac{\overset{21}{\cancel{84}}}{\underset{1}{\cancel{4}}}$ **3rd.** Divide both sides of the equation by 4 so n will be alone on the left side of equation sign.

$n = \$21$ Ans. Dick saved $21.

To solve Type 1 problem, it is much simpler to use an equation method than a proportion method. It requires less computation.

Type 1 Problems include finding the sales tax and the service tip etc.

Solving Type 2 Problems: Finding the Ratio (or Percent) (Read first p.394)

Problem: In a math test, Judy got 32 problems correct out of 40. What percent of correct answers did she get?

Analysis: The problem -- "What percent of 40 (total) is 32 (part)?"
The equation -- n x 40 = 32 and n = 32/40 (32 − 40)
The question asked -- What % of correct answer did Judy get?

Clue: If you know the shortcut (**See p.391**), all you have to do is to divide the numbers and write the answer as a percent by moving the decimal point two places to the right and adding the %. See Method 1.

Method 1. Using Equation - Finding n and then write as n%

$$n\% = \frac{32}{40} = 0.8$$

1st. Divide. In algebra, division is written in a fraction form.

$$n\% = 0.8 \times \frac{100}{100}$$

$$= \frac{80}{100} = 80\%$$

2nd. The question is asking for a ratio. Multiply 0.8 with 100/100 (=1) changing the decimal to a fraction with a denominator of 100. Then write as a percent.
Ans. Judy got 80% correct.

Method 2. Using Equation - Changing the percent to a decimal

$$(0.01)n = \frac{32}{40} = 0.8$$

1st. Change the % to a decimal 0.01 (1/100), since n here stands for n%.

$$\frac{\cancel{(0.01)}n}{\cancel{0.01}} = \frac{0.8}{0\ 01}$$

2nd. Divide both sides by 0.01 and solve for n.

$$n = 80\%$$

Ans. Judy got 80% correct.

Method 3. Using Proportion - Changing the percent to a ratio

$$\frac{n}{100} = \frac{32}{40}$$

1st. Set up a proportion. Make sure each ratio is comparing the part to the total.

$$n \times 40 = 100 \times 32$$

2nd. Find the cross products.

$$\frac{n \times 40}{40} = \frac{\overset{5}{\cancel{100}} \times \overset{16}{\cancel{32}}}{\underset{1}{\cancel{40}\,\cancel{2}}}$$

3rd. Divide both sides by 40 and solve for n. Use cancellation.

$$n = 80\%$$

Ans. Judy got 80% correct.

Solving Type 3 Problems: Finding the Total (or Base) (Read first p.394)

Problem: According to AA High School, 238 graduates of the class of 1993 went on to college. It represents 68% of the total graduates of the year. How many did graduate from AA High School that year?

Analysis: The problem -- "68% of what number is 238?"
The equation -- $.68 \times n = 238$ and $n = 238/.68$ ($238 \div 68\%$)
(In algebra, division is written in fraction form.)
The question asked -- "How many graduated in 1993 from AA Hi?"

Clue: If you rewrite the percent as a decimal, you solve the problem in one step by dividing the part by the ratio. See Method 2.

Method 1. Using Equation - Changing the percent to a fraction.

$n = 238 \div \frac{68}{100}$	**1st.** Change the percent to a fraction.
$n = 238 \div 0.68$	**2nd.** Compute
$= 350$	Ans. 350 graduated.

Method 2. Using Equation - Changing the percent to a decimal

$n = 238 \div 0.68$	**1st.** Change the percent to a decimal and divide 238 by the decimal.
$= 350$	Ans. 350 graduated.

Method 3. Using Proportion - Changing the percent to a ratio

$\frac{17}{25} = \frac{238}{n}$	**1st.** Set up a proportion. Change the percent to a ratio in lowest terms.
$17 \times n = 25 \times 238$	**2nd.** Find the cross products, n on the left of the equation.
$\frac{\cancel{17} \times n}{\cancel{17}} = \frac{25 \times 238}{17}$	**3rd.** Divide both sides of the equation by 17 and solve for n.
$n = 350$	Ans. The total graduates of the class 1993 was 350.

Percents & Money (Review also "Decimals & Money" p.59)

Since a dollar equals 100 cents, we can express it in the following ways:

% of a dollar	In Decimal	In Word	(In Fraction)
1% of a dollar	**$.01**	1 cent	(1/100 of $1.00)
8% of a dollar	**$.08**	8 cents	(8/100 of $1.00)
10% of a dollar	**$.10**	10 cents	(1/10 of $1.00)
25% of a dollar	**$.25**	25 cents	(1/4 of $1.00)
50% of a dollar	**$.50**	50 cents	(1/2 of $1.00)
75% of a dollar	**$.75**	75 cents	(3/4 of $1.00)
100% of a dollar	**$1.00**	100 cents	($1.00)
150% of a dollar	**$1.50**	150 cents	(1 1/2 of $1.00)

Note: "8% sale tax" means you pay an extra 8 cents for every dollar you spend.

Summary

(Ratios, Proportions, & Percents)

- A ratio is a comparison of two numbers or quantities of the same kind of units. A rate is a comparison of two qualities in different kinds of units.
- A proportion is an equation which shows that two ratios are equal. Cross products is a method used to test whether two ratios are really equal.
- Percents can be considered as a special kind of ratio, or fraction, or decimal. As a ratio, percents always compares with a number 100 As a fraction, percents always have the denominator of 100. As a decimal, percents have the place value of "hundredths."
- Percents, decimals, and fractions are all interchangeable. There are six possible ways of interchange among them.
- There are three basic types of percent problems because a percent problem is made up of only three quantities: percentage, total, and part.
- In solving percent problems, read and analyze the problem first. Use the information given to write a mathematical equation. Then the equation is rearranged so that the unknown term is alone on the left side of the equation and the known terms on its right side. Any number with the percent sign (%) should be changed to a decimal, or a fraction before it is used in computation.

Index

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This is a reference book with the following features:

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- ***It is illustrative. Every page explains mainly one concept or one skill.***
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- ***It states the important points repeatedly to call attention of the readers.***

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