the *family* MATH companion

ARITHMETIC—THE FOUNDATION OF MATH

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Part IV. Fraction Operations

A. Introduction

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Common Fractions - Terminating Decimals (See p.249)

Fractions means division. ***By dividing the numerator by the denominator, we change a common fraction to a decimal.*** In general, a common fraction can be changed to either a terminating decimal or a repeating decimal.

A. Terminating Decimals

Common Fractions	Terminating Decimals	Decimal Fractions	Common Fractions
1/2	.5	5/10	1/2
1/4	.25	25/100	1/4
1/5	.2	2/10	1/2
3/4	.75	75/100	3/4

divide **equal** **reduce**

Clue: If the denominator of the fraction ***divides evenly into 10 or 100,*** the fraction can be changed easily to decimal or decimal fraction.

Example: 2, 5, divide evenly into 10, 100; 4 divides evenly into 100, because **2 and 5 are factors of 10, 100; 4 is a factor of 100.**

Common Fractions - Repeating Decimals (See p.250)

B. Repeating Decimals

				(a)		(b)		(c)
1/3	=	1 ÷ 3	=	.33...,	or	$.3\overline{3}$,	or	.33 1/3
2/3	=	2 ÷ 3	=	.66...,	or	$.6\overline{6}$,	or	.66 2/3
1/6	=	1 ÷ 6	=	.16...,	or	$.16\overline{6}$,	or	.16 2/3

In dividing the numerator by the denominator, sometimes the division never comes out evenly no matter how many times we divide. **If a digit or a group of digits repeats itself,** we call it a repeating decimal. A repeating decimals can be expressed in one of the following ways:

(a) By three dots - Indicates that the **last digit** repeats without end.
Read the three dots as *"and so on."*
(b) By a bar - Indicates that **the digit(s) under the bar** repeats endlessly.
(c) By a fraction - Write the remainder over the divisor.
Read .33 1/3 as *"thirty-three and one-third hundredths."*
(d) By rounding - Find the quotient **one more place than required** and round off.

Clue: If the denominator of the fraction *does not* **divide evenly into 10, 100,** the fraction will be a repeating decimal.

Estimating Fractions (Review first "Rounding" p.30)

Rounding is the method we use in estimating the sums and differences of fractions including mixed numbers. We compare the numerator and the denominator of the given fraction and round it either to 0, or 1/2, or 1 as described in the following:

Examples:	**Compare:**	**Rounded to:**
$\frac{1}{8}, \frac{1}{25}, \frac{3}{64} \quad \frac{5}{72},$	The numerator is much smaller than the denominator.	**0**

Fractions that are less than 1/2 are rounded down to 0 because it is closer to 0 on a number line.(See p.31)

$\frac{3}{8}, \frac{4}{9}, \frac{7}{13}, \frac{13}{25},$	The numerator is about half of the denominator.	$\mathbf{\frac{1}{2}}$

Multiplying the numerator by 2, if the product is close to the denominator, round to 1/2.

$\frac{5}{6}, \frac{7}{8}, \frac{10}{12}, \frac{28}{30},$	The numerator is about the same as the denominator.	**1**

Guideline: If a fraction is less than 1/2, round down to 0. If it is greater than 1/2, round up to 1.

Estimating Sums and Differences of Fractions and Mixed Numbers **(Read p.230)**

a) Estimate 3 1/7 + 5 8/9 + 4/5 (≈ means **"about"** or **"is approximate to"**)

3 + 5 = 8 **Step 1.** Add the whole number parts.

$\frac{1}{7} \approx 0 \quad \frac{8}{9} \approx 1 \quad \frac{4}{5} \approx 1$ **Step 2. Round** the fractional parts and estimate the sum: 2.

8 + 2 = **10** **Step 3.** Add the two sums: **about 10.**

b) Estimate 9 4/7 - 1 7/8 (The symbol means **"therefore"**)

9 - 1 = 8 **Step 1.** Subtract the whole number part.

Since 4/7 ≈ 1/2, 7/8 ≈ 1
4/7 < 7/8

∴ 9 4/7 - 1 7/8 **< 8**

Step 2. Adjust the above difference by comparing the **fractional parts.** Since 4/7 is less than 7/8, the estimated difference will be **less than 8.**

Note: A **simpler way** to estimate the sums and differences of **mixed numbers** is to round the fractional part to the nearest whole number and then add or subtract.

Estimating Products of Fractions

We can estimate the products of fractions and mixed numbers either by **rounding** or by **using compatible numbers.**

a) Estimate 7 1/2 x 5 5/6

$$\begin{array}{rcr} 7\ 1/2 & \longrightarrow & 7 \\ \underline{\times\ 5\ 5/6} & \longrightarrow & \underline{\times\ 6} \\ & & 42 \end{array}$$

Using Rounding **(Review first p.232)**
1st. Round the fractional part in each factor to the nearest whole number.
2nd. Multiply the **rounded factors: about 42.**

b) Estimate 33 2/9 x 16 8/9

$$\begin{array}{lccccc} & 33\ 2/9 & \times & 16\ 8/9 & & \\ & \downarrow & & \downarrow & & \\ & 30 & \times & 20 & = & 600 \\ \text{or} & 30 & \times & 15 & = & 450 \end{array}$$

Using Compatible Numbers
Substitute the given factors with a pair of numbers that are **close in value,** which you can **multiply mentally.** Often there are more than one pair of compatible numbers.

Note: The actual product is between 450 and 600. However, the first estimated product 600 is closer to the actual product because one factor was rounded down and the other factor rounded up.

Estimating Quotients of Fractions

Estimate the quotients of fractions in the same way you estimate the products of fractions.

a) Estimate 11 12/13 ÷ 3 7/9.

$$\begin{array}{cccccc} 11\ 12/13 & \div & 3\ 7/9 & & \\ \downarrow & & \downarrow & & \\ 12 & \div & 4 & = & 3 \end{array}$$

Using Rounding **(See p.233)**
Round the fraction to the nearest whole number and divide.

b) Estimate 19 1/7 - 4 3/11

$$\begin{array}{cccccc} 19\ 1/7 & \div & 4\ 3/11 & & \\ \downarrow & & \downarrow & & \\ 20 & \div & 4 & = & 5 \end{array}$$

Using Compatible Numbers
Since 19 **can not be divided** by 4, rounding is not the method to use. **Replace** both numbers with a pair of numbers, such as 5 x 4 = 20, **that can divide easily.**

Remember: When using compatible numbers, **choose the numbers that are close to the original numbers,** so that the estimated product or quotient will be closer to the actual product or quotient.

Prior Knowledge For Working With Fractions

Many students got lost in the maze of fractions because they didn't realize that fractions, unlike whole numbers and decimals, requires the **prior knowledge of many concepts and skills.** The check (✓) in the following chart shows you the concepts and skills that are needed to do each operation:

Concepts & Skills	Addition	Subtraction	Multiplication	Division
* Multiplication and Division facts	✓	✓	✓	✓
* Finding LCM/LCD	✓	✓		
* "1" in fraction form	✓	✓	✓	✓
* Writing equivalent fractions	✓	✓	✓	✓
* Borrowing		✓		

Note: Multiplication/Division facts are absolutely essential in working with fractions.

Concepts & Skills	**Addition**	**Subtraction**	**Multiplication**	**Division**
* Changing mixed numbers to improper fractions			✓	✓
* Writing whole numbers as fractions			✓	✓
* Writing the reciprocal of a number				✓
* Cancellation			✓	✓
* Changing improper fractions to mixed numbers	✓	✓	✓	✓
* Finding GCF	✓	✓	✓	✓
* Reduce fractions to lowest terms	✓	✓	✓	✓

Note: It is based on "1 property of multiplication/division" that we write equivalent fractions, do cancellation, etc.

Reviewing The Basic Concepts & Skills

* To Change A Mixed Number to An Improper Fraction (See p.95)

Remember· A mixed number = whole number + fraction (with "+" sign omitted)

(mixed number) — 5 1/3 = $\frac{(5 \times 3) + 1}{3}$ = $\frac{16}{3}$ — (improper fraction)

$$\uparrow$$

$$\boxed{\frac{\textbf{(whole number x denominator) + numerator}}{\textbf{denominator}}}$$

* To Change An Improper Fraction to A Mixed Number (See p.94)

(improper fraction) $\frac{16}{3}$ = $16 \div 3$ = 5 1/3 (mixed number)

$$\uparrow$$

$$\boxed{\textbf{numerator} \div \textbf{denominator}}$$

* To Write The Reciprocal Of a Number is to Invert the number (See p.326)

$$\begin{tikzcd} 3 \arrow[dr] & 5 \\ 5 \arrow[ur] & 3 \end{tikzcd}$$

(fraction) $\frac{3}{5}$ (Reciprocal of 3/5)

(To invert means to switch the top and the bottom.)

* To write "1" as a fraction (See p.299)

$1 = \frac{3}{3} = \frac{5}{5} = \frac{3/4}{3/4} \ldots$

Use **the same number** for the numerator and the denominator.

* To Write A Whole Number as A Fraction (See p.298)

$4 = \frac{4}{1}, \quad 7 = \frac{7}{1},$

Use the whole number as the numerator and 1 as **the denominator.**

* To Write An Equivalent Fraction of Higher Terms (See p.301)

$\frac{2}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9}$

Multiply the numerator and denominator of the fraction **by the same non-zero number. 3/3 = 1.**

* To Reduce A Fraction To Lowest Terms (See p.306)

$\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$

Divide the numerator and denominator of the fraction **by the largest number (GCF)** that ***divide evenly*** into both numbers.

Remember, fractions are always written in lowest terms.

Summary

(Introduction)

* Fractions means division. By dividing the numerator (top number) by the denominator (bottom number), we change a common fraction, or any fraction, to a decimal. If the denominator of a common fraction is a factor of 10 or 100, the decimal will be a terminating decimal, if it is not a factor of 10, or 100, the decimal will be a repeating decimal.

* A repeating decimal can be expressed (a) by three dots, (b) by a bar, (c) by a fraction, or (d) by rounding off to the desired place. Fraction form is the best way to express a repeating decimal.

* To avoid computational error, estimate the answer first. Then compare the answer with the original estimation.

* Any whole number can be written as a fraction by using the number as the numerator and "1" as the denominator To write "1" as a fraction, use any non-zero number for both the numerator and the denominator

Part IV. Fraction Operations

B. Factors & Multiples

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GCF (Greatest Common Factor) vs. LCM (Least Common Multiple)

Students often confuse GCF with LCM. Read the following lists side by side so that you see the differences between the two.

GCF has to do with:	**LCM** has to do with:
- Reducing a fraction to **lowest terms.**	- Changing **unlike** fractions to like fractions.
- **One** fraction.	- **Two or more** fractions
- The **numerator** and the **denominator** of a fraction.	- The **denominators** of unlike fractions
- Sums, differences, products, & quotients.	- Adding, subtracting, and comparing unlike fractions.
- Writing equivalent fraction **by division.**	- Writing equivalent fraction **by multiplication.**

GCF & Reducing A Fraction to Lowest Terms (Read first "GCF vs. LCM" p.280)

It is required that a fraction must be reduced to lowest terms at the end of computation. In order to reduce a fraction to lowest terms,

1st. You must know how to find the GCF of its numerator and denominator.
You will be able to find the GCF of two numbers,
a) if you know **what factors are** and how to find them. (pp.282,285)
b) if you know **factoring and prime factorization** (p.295)

2nd. You must know how to use the GCF to reduce the fraction.
You will be able to reduce a fraction,
a) if you know the **"1 property of division"** (p.298) and
b) if you know how to write **"1" in fraction** (p.299).

3rd. You must know when a fraction is in lowest terms.
You know when a fraction is in lowest terms,
a) if you know **prime and composite** numbers (p.290).
b) if you understand the meaning of **"relative prime"** (p.293).

Connection: It is important that you learn well factoring, prime factorization, & the concept of GCF. These skills are also used in Algebra to factor algebraic expressions, polynomials, quadratic equations, etc.

Factors - Factors & Divisors (See also "Divisibility Rules" p.288)

$$\begin{array}{ccccc} \text{a factor} & & \text{a factor} & & \text{product} \\ 3 & \text{x} & 4 & = & 12 \end{array}$$

> A number has a "limited" number of factors. And 1 is a factor of every number

$$\begin{array}{l} \text{factor} \\ \downarrow \\ \begin{array}{r} 4 \\ 3\overline{)\,12} \\ -\,12 \\ \hline 0 \end{array} \end{array} \qquad \begin{array}{l} \text{factor} \\ \downarrow \\ \begin{array}{r} 3 \\ 4\overline{)\,12} \\ -\,12 \\ \hline 0 \end{array} \end{array} \qquad \begin{array}{r} 2 \\ 5\overline{)\,12} \\ -\,10 \\ \hline 2 \leftarrow \text{remainder} \end{array}$$

Definition of Factors:

* Factors are numbers that multiply to give a product. We say, **"3 (or 4) is a factor of 12"** because 12 has 6 factors (1, 2, 3, 4, 6, 12) and 3 (or 4) is only one of them.

* Factors are *exact divisors* that divide evenly into a number. We say, **"3 (or 4) is *an exact divisor* of 12**, and also ***a factor* of 12"** because **division and multiplication are *inverse operations:***

12 - 3 (divisor) = 4 (quotient) **and** 4 (factor) x 3 (factor) = 12

This shows that we can use *division* to find the factors of a number.

Multiples - Multiples & Products (See also p.175)

$$\overset{\text{factor}}{3} \quad \text{x} \quad \overset{\text{factor}}{4} \quad = \quad \overset{\text{product}}{12}$$

A number has an "unlimited" number of multiples. And 0 is a multiple of every number

multiples of 3

$3 \text{ x } 0 = 0$
$3 \text{ x } 1 = 3$
$3 \text{ x } 2 = 6$
$3 \text{ x } 3 = 9$
$\boxed{3 \text{ x } 4 = 12}$
⋮

multiples of 4

$4 \text{ x } 0 = 0$
$4 \text{ x } 1 = 4$
$4 \text{ x } 2 = 8$
$\boxed{4 \text{ x } 3 = 12}$
⋮

Definition of Multiples:

* A multiple of a number (say 3) is the product of that number (3) and any whole number (0, 1, 2, 3, ...). So, multiple is another name for product.

We say, "12 is ***a*** multiple of 3 (or 4)" because 3 (or 4) has many multiples and 12 is ***one*** of them. And

We say, "12 is ***a common multiple*** of 3 and 4" because some numbers have ***multiples in common.***

Remember: A product has to do with at least a pair of numbers - **two numbers;**
A product is a multiple of either number - **one number.**

Relating Factors, Divisors, Multiples, & Divisibility (Review first p 282,283)

The following example shows how Factors, Divisors, Multiples, and Divisibility are related. **A knowledge you need to have in working with fractions.**

5 is a **factor** of 15.
3 is a **factor** of 15.

A number has a limited number of factors. 15 has only 4 factors.

3 is a **divisor** of 15.
5 is a **divisor** of 15.

3 x 5 = 15

15 is a **multiple** of 5.
15 is a **multiple** of 3.

A number has an unlimited number of multiples. The multiples of 3 (or 5) are endless.

15 is ***divisible*** by 3.
15 is ***divisible*** by 5.

Remember the following statements:

a) **A number is a multiple of each of its factors/divisors.**

Example: 15 is a multiple of its factor/divisor.

b) **A number is divisible (can be divided) by another number if the second number is a factor or divisor of the first number.**

Example: 15 is divisible by 5 because 5 is a factor/divisor of 15.

Finding Factors: A How-To (Review first "Factors" p.282)

How To Find the Factors of 36:

$$
\begin{aligned}
36 &= \overset{\substack{\text{factor}\\ \downarrow}}{1} \text{ x } \overset{\substack{\text{factor}\\ \downarrow}}{36}\\
&= 2 \text{ x } 18\\
&= 3 \text{ x } 12\\
&= 4 \text{ x } 9\\
&= 6 \text{ x } 6
\end{aligned}
$$

Method 1. Using Multiplication

Factors are numbers that you multiply together to give a product. List all the **pairs of numbers** when multiplied gives the product of 36. The factors of 36 are:

1, 2, 3, 4, 6, 9, 12, 18, 36

Remember:

* A number has a **limited** number of factors.

* **Each pair** of factors listed above is **a factor form** of 36.

* And 1 x 36, 2 x 18, 3 x 12, 4 x 9, 6 x 6, are **all** the factor forms of 36.

Note: We actually use **division** to find the pair of factors such as 1 x 36, 2 x 18, 3 x 12, because we remember only the one-digit multiplication facts.

Finding Factors: A How-To

How To Find the Factors of 36:

	divisor		quotient
36 ÷	1	=	36
36 -	2	=	18
36 -	3	=	12
36 ÷	4	=	9
36 ÷	5	=	7 r1
36 ÷	6	=	6
36 ÷	7	=	5 r1
36 ÷	8	=	4 r4
36 -	9	=	4

Method 2. Using Division

Divide 36 by 1, 2,... **the natural** numbers. The **exact divisors** and **their quotients** are the factors of 36 because division and multiplication are ***inverse operations:***

36 - 4 = 9, and
4 x 9 = 36

Therefore, the factors of 36 are:
1, 36, 2, 18, 3, 12, 4, 9, 6

Note: 5, 7, 8 are divisors but not factors of 36.

Note: We do not have to try any whole number greater than 6 as a divisor, since factors begin to repeat after 36 - 7 = 5 r1.

How to Find the Factors of 36:

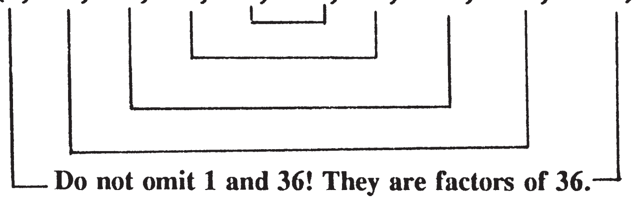
Method 3. This method is the same as **Method 2** except the way the factors are listed - from the smallest to the largest. Here is a how-to:

1st. Write 36 and a set of parentheses large enough to list the factors of 36: 36 (   ).

2nd. Start from the smallest divisor. Write it at the left end and its quotient at the other end: 36 (1,   36)

3rd. Then the next smallest divisor and its quotient. Continue in this manner until all factors of 36 are listed:

36 (1, 2, 3, 4, 6, 6, 9, 12, 18, 36)



Do not omit 1 and 36! They are factors of 36.

Finding Factors By Using Divisibility Rules (Review first p.282)

Divisibility or divisible means **"that a number can be *divided evenly* with *no remainder."*** It refers to **exact divisor** and **its quotient** because division is a common way of finding the factors of a number

Divisibility rules help us to know, by simple inspection without dividing, **whether a number can be divided evenly by a certain number.** They are useful especially in dealing with large numbers.

* **A number is divisible by 2, if its last digit is 2, 4, 6, 8, or 0.**
Example: 4, 36, 74, 308, 15790,... each number has a factor of 2.

Remember: Any number whose last digit is 2, 4, 6, 8, 0, is an **even number.** They are multiples of 2.

* **A number is divisible by 5, if its last digit is 0 or 5.**
Example: 5, 45, 95, 100,... each number has a factor of 5.

* **A number is divisible by 10, if its last digit is 0.**
Example: 10, 400, 2,000,... each number has a factor of 10.

* **A number is divisible by 4, if the <u>last 2 digits</u> of the number are zeros or is divisible by 4.**
Example: 536: 36 ÷ 4 = 9, 536 is divisible by 4. 4 is a factor.

* **A number is divisible by 8, if the <u>last three digits</u> of the number are zeros or is divisible by 8.**
Example: 5384: 384 ÷ 8 = 48; 5384 is divisible by 8. 8 is a factor.

* **A number is divisible by 3, if the <u>sum of the digits</u> of the number is divisible by 3.**
Example: 705: 7 + 0 + 5 = 12; 12 ÷ 3 = 4
705 is divisible by 3. 3 is a factor.

* **A number is divisible by 9, if the <u>sum of the digits</u> of the number is divisible by 9.**
Example: 1697: 1 + 6 + 9 + 7 = 23; 23 ÷ 9 = 2 **R5**.
1697 is not divisible by 9. 9 is not a factor.

* **A number is divisible by 6, if the number is divisible by 2 and also by 3.**
Example: 1422: 2 ÷ 2 = 1 (divisible by 2)
1422: 1 + 4 + 2 + 2 = 9; 9 ÷ 3 = 3 (divisible by 3)
Therefore, 1422 is divisible by 6. 6 is a factor.

Connection: These rules are helpful in finding factors and also in factoring.

Factors - Prime & Composite Numbers (See also p.10)

When we examine the factors of say the first 11 natural numbers, we observe the characteristic of two groups of numbers as seen below:

Number	List of Factors	Number of Factors
1	1	1
2	1, 2	2
3	1, 3	2
4	1, 2, 4	3
5	1, 5	2
6	1, 2, 3, 6	4
7	1, 7	2
8	1, 2, 4, 8	4
9	1, 3, 9	3
10	1, 2, 5, 10	4
11	1, 11	2

Observations

* All numbers, except 1, have at least 2 factors.

* The numbers 2, 3, 5, 7, 11, have **only 2 factors** **1 and the number itself.** We call these numbers **Prime Numbers.**

* The numbers which have **more than 2 factors** are called **Composite Numbers.**

We do not know how many prime numbers there are because there is no way to predict when it will occur. However, we can use Eratosthenes' method to sort out the prime numbers up to certain numbers. See next page.

Finding the Prime Numbers (Compare with "Even & Odd Numbers" p.9)

	②	③	~~4~~	⑤	~~6~~	⑦	~~8~~	~~9~~	~~10~~
⑪	~~12~~	⑬	~~14~~	~~15~~	~~16~~	⑰	~~18~~	⑲	~~20~~
~~21~~	~~22~~	㉓	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	㉙	~~30~~
㉛	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	㊲	~~38~~	~~39~~	~~40~~
㊶	~~42~~	㊸	~~44~~	~~45~~	~~46~~	㊼	~~48~~	~~49~~	~~50~~

The Sieve of Eratosthenes

Around 200 B.C. Eratosthenes, a Greek mathematician, developed the following method to sort out the prime numbers.

First, write the whole numbers up to 50 or 100 and then follow the step described below:

Step 1. Circle 2. Then cross out all the multiples of 2 which are the numbers that can be divided evenly by 2.

Step 2. Circle 3, the next number not crossed out. Again, cross out all the multiples of 3. **Some numbers have multiples in common.**

Step 3. Circle 5, the next number not crossed out. Cross out all the multiples of 5.

Step 4. Continue the procedure until all numbers are either circled or crossed out. **The circled numbers are prime** and the crossed out numbers are composite.

Note: 1 is omitted because 1 is neither prime nor composite. It is a set by itself.

Prime Numbers vs. Composite Numbers (See also p.10)

The set of natural numbers is made up of three sets of numbers: the set of 1 by itself, the set of prime numbers, and the set of composite numbers.

Definition of Prime Number:

* A number is a prime number if it has **only two factors - 1 and itself.**
* A number is a prime number if it is **divisible only by 1 and itself.**
* A number is a prime number if it can be written **as a product** of two different numbers in **only one way, the order may differ.**
Example: 5 = 1 x 5 or 5 x 1 (Differ only in order)

Definition of Composite Number:

* A number is a composite number if it has other **factor(s) in addition to 1 and itself.**
* A number is a composite number if it is **divisible by a natural number other than 1 and itself.**
* A number is a composite number if it can be written **as a product** of two different natural numbers in **more than one way.**
Example: 12 = 1 x 12 & 12 = 2 x 6 & 12 = 3 x 4

Relatively Prime Numbers And Lowest Terms

Relatively Prime Numbers:

Two numbers are said to be relatively primes if **1 is the only common factor.** The GCF or the GCD of the numbers is 1. For example:

4 and 9 are relatively prime.

The factors of 4: **1**, 2, 4	*The only* common factor
The factors of 9: **1**, 3, 9	of 4 and 9 is **1**.

Two numbers are relatively prime even if one or both are composite like 4 and 9; 15 and 16.

Lowest Terms or Simplest Form:

A fraction is said to be in lowest terms **when its numerator and denominator are relatively prime**, which means when their GCF is 1. For example:

4/9 is in lowest terms since 4 and 9 are relatively prime.

To reduce a fraction to lowest terms, we divide its numerator and denominator by the greatest common factor (GCF). That is why GCF is also called the greatest common divisor (GCD).

Factor & Factoring (Review first "Factors" p 282)

Factor(s):

$$\overset{\text{factor}}{3} \times \overset{\text{factor}}{6} = 18$$

* Factor as a noun refers to **the numbers** when multiplied give the product. The factors of 18 are **1, 2, 3, 6, 9, 18.**

Factor, Factoring:

$$\begin{aligned} 18 &= 3 \times 6 \\ &= 2 \times 9 \\ &= 1 \times 18 \end{aligned}$$

Note: 2 & 3 are prime factors ; 6, 9, 18 are composite factors

* Factor as a verb refers to **the process** of writing a number as a product of any two of its factors, prime or composite. We say:

- **"18 has been factored."** And
- **"3 x 6 is *a* factor form of 18"** **because 18 has other factor forms.**

Connection: Factoring is an important process in mathematics, especially in Algebra.

Prime Factoring, Prime Factorization (Review "Prime & Composite" p.292)

Prime Numbers:

2, 3, 5, 7, 11

* Prime refers to **the numbers** that have **only two factors**. Memorize the **first 5 prime numbers** on the left.

Prime Factoring, Prime Factorization:

$$18 = \overset{\text{factoring}}{\underline{3 \text{ x } 6}} = \overset{\text{prime factoring}}{\boxed{3 \text{ x } 2 \text{ x } 3}} \qquad ①$$

$$18 = \underline{2 \text{ x } 9} = \boxed{2 \text{ x } 3 \text{ x } 3} \qquad ②$$

* Prime factoring is **the process** of writing a **composite** number as the product of **prime** factors.

* Prime factorization is **the result** of prime factoring. We say, **"18 has been factored into prime numbers."** **"3 x 3 x 2 is the prime factorization of 18."**

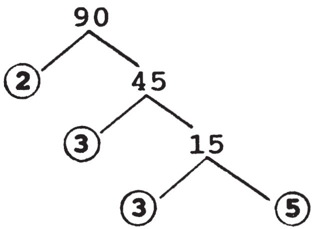
Note: Each composite number has **"only one set of prime factors"** – **the factors may be in different order** as in ① & ② above. We can factor any composite number into prime factors either by **"Factor Tree"** or by **"Invert Short Division"** **(See next pages).**

Prime Factorization: Factor Tree Method (Review first "Prime & Composite" p.292)

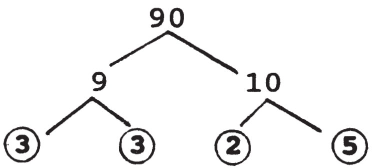
To Factor 90 into Prime Numbers by Using The Factor Tree:

Write 90 as the **product of two of its factors.** Write one on the left and the other on the right under the number 90. Continue to write **composite** numbers as the products of two factors until **all factors are prime.**

a)



b)



Note: The prime factors of 90 are the same for both a) and b).

The prime factorization of 90 is 2 x 3 x 3 x 5.

The **Fundamental Theorem of Arithmetic** states that **every composite number has** ***exactly one prime factorization*** **though the order of the factors may differ.**

Prime Factorization: Invert Short Division Method (See also "GCF Method" p.306)

Prime factorization is one of the methods used to find the GCF of two (or more) numbers. GCF is used in reducing a fraction to lowest terms.

To Find the GCF of 54 and 90 by Using Prime Factorization:

Step 1. **Factor each number into prime factors.** Divide each number by **its own prime** factor starting with *smallest prime number.* Continue the process until the **last number (quotient) is prime.**

prime →	**2**) 54	**prime →**	**2**) 90
prime →	**3**) 27	**prime →**	**3**) 45
prime →	**3**) 9	**prime →**	**3**) 15
prime →	**3**	**prime →**	**5**

Step 2. **Write each number as the product of prime factors.**

The prime factorization of 54: 2 x 3 x 3 x 3
The prime factorization of 90: 2 x 3 x 3 x 5

Step 3. **Find the GCF of two numbers.** The GCF is the product of the factors which appear in both prime factorizations.

GCF of 54 and 90: 2 x 3 x 3 = 18

"1" Property of Multiplication & division (Review first p.121)

It is based on the following two properties that we write all equivalent fractions. Read this page and next page together.

"1" property of multiplication:

8 x 1 = 8 The property says, **"Any number multiplied by 1 is the number."**

3/7 x 1 = 3/7 The property applies also to fractions.

"1" Property of Division:

15 ÷ 1 = 15 The property says, **"Any number divided by 1 is the number."**

8/9 ÷ 1 = 8/9 Again it applies also to fractions.

Writing Whole Numbers As Fractions - Any whole number can be changed to a fraction by **using the number as the numerator** and **1 as the denominator.**
Example: 5 = 5/1; 12 = 12/1, etc.

Writing "1" in Fraction Form

To use "1" properties (see opposite page) in fractions, we write 1 as improper fractions **with the same number for both numerator and denominator (0 excluded)** because "Any number divided by itself equals 1" (p.196) That means we can write 1 as fractions in *many ways* - either as simple fractions or as complex fractions.

* Write 1 As Simple Fractions (See p.92)

$$1 = \frac{2}{2} = \frac{25}{25} = \frac{100}{100}, \ldots$$

To multiply/divide a fraction by 2/2, is the same as to multiply/divide it by 1. **The value of the fraction is not changed.**

Note: This knowledge is needed in subtracting fractions with borrowing.

* Write 1 As Complex Fractions

$$1 = \frac{\;\frac{2}{3}\;}{\frac{2}{3}} = \frac{\;\frac{5}{1}\;}{\frac{5}{1}} = \frac{\;\frac{9}{4}\;}{\frac{9}{4}}, \ldots$$

By applying "1" property of multiplication with 1 written as complex fraction, we change division of fraction to multiplication (See p.327, 365)

Connection: "1" properties with 1 written in fractions are used in writing equivalent fractions (p.300), in cancellation (p.304), etc.

Writing Equivalent Fractions : A How-To

By using the "1" **property of multiplication/division with** *1 written as fractions* (Review also last two pages), we can write fractions of **equal value** to replace the given fraction. Here is how we do it:

To Write Equivalent Fraction With Lower Terms:

1 in fraction

6/9, 4/6, 2/3 are all equivalent fractions of 12/18

$$\frac{12}{18} = \frac{12}{18} \div \frac{2}{2} = \frac{12 \div 2}{18 \div 2} = \frac{6}{9}$$

$$\frac{12}{18} = \frac{12}{18} - \frac{3}{3} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}$$

$$\frac{12}{18} = \frac{12}{18} - \frac{6}{6} = \frac{12 - 6}{18 - 6} = \frac{2}{3}$$

To write equivalent fraction with lower terms, ***divide*** the numerator and the denominator of a given fraction by their common factors.

To divide the fraction, 12/18, by 2/2, 3/3 etc. is the same as to divide the fraction by 1.

Connection: ***By dividing*** a fraction with the GCF, we reduce the fraction to lowest terms.

To Write Equivalent Fraction With Higher Terms:

1 in fraction

4/6, 6/9, 10/15,... are equivalent fractions of 2/3.

$$\begin{aligned}\frac{2}{3} &= \frac{2}{3} \times \frac{2}{2} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6}\\ &= \frac{2}{3} \times \frac{3}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9}\\ &= \frac{2}{3} \times \frac{5}{5} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}\\ &\quad\vdots \qquad \vdots \qquad\quad \vdots \qquad\quad \vdots\end{aligned}$$

To write equivalent fractions with higher terms, ***multiply*** the numerator and the denominator of the given (original) fraction with the **same non-zero number:** 2/2, 3/3, etc.

To multiply the fraction by 2/2, etc. is the same as to multiply it by 1.

Remember: In general, we can ***multiply*** the numerator and the denominator of a fraction **by any non-zero number;** but they can be ***divided only*** **by common factors.**

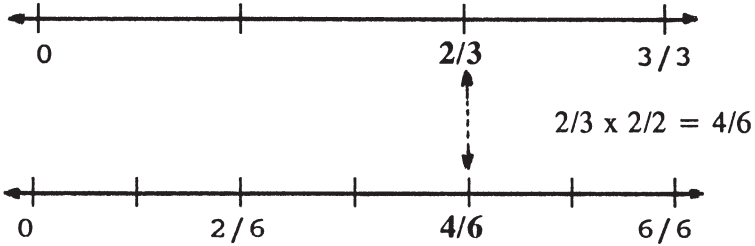
Connection: *By multiplying,* we change fractions of different denominators to fractions of the same denominator using the LCM. **(See p.311)**

Test Of Equivalent Fractions

To write equivalent fractions, we **either multiply or divide** the given fraction by **a fraction which equals to 1** -- 2/2, 3/3, 4/4,... (p.300)
To test whether or not two fractions are equivalent, we can use one of the following methods:

1. Use A Number Line

If two fractions are equal, they will have the **same point** on the number line. Let's locate the point for 2/3 and 4/6 on the number lines to see if they are equivalent fractions:



Divide 1 into 3 equal parts, and locate 2/3 on the line.

Divide 1 into 6 equal parts, and locate 4/6 on the line.

We know that 2/3 and 4/6 **are equivalent fractions** because they share the same **point** on the number line.

2. Cross Products or Cross Multiplication (See also "Proportion" p.375)

To determine whether or not 3/7 and 12/28 are equivalent:

$$\frac{3}{7} \overset{?}{\times} \frac{12}{28}$$

1st. **Multiply** the denominator of the first fraction by the numerator of the second:

$7 \times 12 = \mathbf{84}$

2nd. **Multiply** the numerator of the first fraction by the denominator of the second:

$3 \times 28 = \mathbf{84}$

$$\frac{3}{7} = \frac{12}{28}$$

3rd. If the product of the first one ***equals*** the product of the second one, the two fractions are equivalent: 84 = 84.

Note: Do you see why the method is called "cross products"?

3. Compare Fractions in Lowest Terms

To decide whether or not 14/21 & 24/40 are equivalent:

$$\frac{14}{21} = \frac{14 - 7}{21 \div 7} = \frac{2}{3}$$

$$\frac{24}{40} = \frac{24 - 8}{40 \div 8} = \frac{3}{5}$$

1st. Reduce each fraction to lowest terms using **any method** given on pages 306-309.

2nd. Compare the fractions in lowest terms to see if they are identical. If not, the two fractions are not equivalent.

Cancellation - Cancelling Factors

To cancel means ***to divide*** **(not subtract).** Cancellation is used to simplify computation by **eliminating common factors** before multiplying. It utilizes the **"1" property of division** - the concept that lies behind the following computation:

1. In Multiplication of Fractions: 27/12 x 2/9 x 4/3

$$\frac{\overset{\overset{1}{\cancel{3}}}{\cancel{27}} \times \overset{1}{\cancel{2}} \times \overset{2}{\cancel{4}}}{\underset{\underset{3}{\cancel{6}}}{\cancel{12}} \times \underset{1}{\cancel{9}} \times \underset{1}{\cancel{3}}} = \frac{2}{3}$$

① Divide 12 & 2 by 2 → 6 & 1
② Divide 6 & 4 by 2 → 3 & 2
③ Divide 27 & 9 by 9 → 3 & 1
④ Divide 3 & 3 by 3 → 1 & 1

You can cancel factors in any order, as long as the numerator and the denominator has a common factor Knowing the "Divisibility Rules" would help you to find factors (p.288-289)

2. In Reciprocals: 5/7 x 7/5 = 1

$$\frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{7}}} \times \frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{5}}} = 1$$

or

$$\frac{5}{7} \times \frac{7}{5} = \frac{35}{35} = 1$$

(The fraction 35/35 = 1)

Use either cancellation or multiplication, the product of the reciprocals is always 1.

3. In Reducing a Fraction to Lowest Terms: `24/54`

$$\frac{24}{54} \overset{\div 2}{\underset{\div 2}{=}} \frac{12}{27} \overset{\div 3}{\underset{\div 3}{=}} \frac{4}{9} \quad \text{or} \quad \frac{24 = 2 \text{ x } 3 \text{ x } 4}{54 = 2 \text{ x } 3 \text{ x } 9} = \frac{4}{9}$$

Note: "Successive Deduction" (p.308) and "Prime Factorization" (p.309) utilize cancellation.

4. In Division of Whole Numbers: `Divide 720 by 48`

$$\frac{720 = \cancel{2} \text{ x } 3 \text{ x } \cancel{4} \text{ x } 5 \text{ x } \cancel{6}}{48 = \cancel{2} \text{ x } \cancel{4} \text{ x } \cancel{6}} = 3 \text{ x } 5 = 15$$

The answer is 15.

Write as a fraction. Factored and then cancel. Using long division gives the same answer

Remember:

$\frac{3 \text{ x } \cancel{7}}{\cancel{7}}$	**You can cancel the 7's, the common factor, because multiplication (3 x 7) and division (1/7) are inverse operations.**
$\frac{3 + 7}{7}$	**You can not cancel the 7's because 7 is not a common factor Addition (3 + 7) and division (1/7) are not inverse operations.**
$\frac{37}{7}$	**You can not cancel the 7 in numerator and the 7 in denominator, because they are digits, not factors.**

Reducing A Fraction To Lowest Terms: GCF Method

To Reduce 24/36 by Common Factors, GCF:

Step 1. List **all** the factors of 24 & 36. **(See Method 3 on p.287)**

24 (**1**, **2**, **3**, **4**, **6**, 8, **12**, 24)
36 (**1**, **2**, **3**, **4**, **6**, 6, 9, **12**, 18, 36)

Step 2. List the factors that are **common** to both lists.

The common factors of 24 and 36 are:
1, 2, 3, 4, 6, **12**

Step 3. Choose the **greatest common factor** (GCF), the **largest number** found in the above list of the common factors.

The GCF of 24 and 36 is **12.**

Step 4. **Divide** 24 & 36 **by 12**, their GCF, respectively.

$$\frac{24 \div 12}{36 \div 12} = \frac{2}{3}$$

The lowest terms of 24/36.

Note: See comment on this method on p.310.

Reducing A Fraction To Lowest Terms: Invert Short Division Method

To Reduce 48/72 by Invert Short Division:

Step 1. Write 48 & 72 side by side as below, **the numerator first.** Divide both numbers by **any common factor.** Continue the process until the numbers are **relatively prime.**

common factor

composite

$$\begin{array}{r|rr} 2 & 48 & 72 \\ \hline 2 & 24 & 36 \\ \hline 6 & 12 & 18 \\ \hline & 2 & 3 \end{array}$$

48/72

2/3 ← **lowest terms**

2 & 3 are relative prime.

Step 2. Multiply the common factors to get GCF of 48 & 72.
The GCF/GCD of 48 & 72: **2 x 2 x 6 = 24**

Step 3. Divide 48/72 by their GCF 24 respectively.

$$\frac{48 \div \mathbf{24}}{72 \div \mathbf{24}} = \frac{\mathbf{2}}{\mathbf{3}}$$

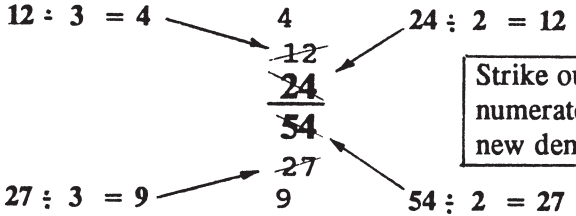
2/3, the lowest terms of 48/72.

Note: To reduce a fraction to lowest terms, Steps 1 is all you need.

Reducing A Fraction To Lowest Terms: Successive Deduction Method

To Reduce 24/54 by Successive Deduction:

Divide the numerator and the denominator by **obvious common factor.** Continue to divide the new numerator and denominator by common factor until they are **relatively prime.**



Strike out 24 & 54. Write 12, the new numerator, above 24 and 27, the new denominator, below 54. Repeat.

Each time you divide 24/54 by a common factor, you are writing **an equivalent fraction** of 24/54 in lower terms:

$$\frac{24}{54} = \frac{12}{27} = \frac{4}{9}$$

So, The lowest terms of 24/54 is 4/9. (4 & 9 are relatively primes.)

Note: This method is similar to Invert Short Division.

Reducing A Fraction to Lowest Terms: Prime Factorization Method (see p 295)

To Reduce 48/54 by Prime Factorization:

Step 1. Factor each number into primes numbers. Methods given on p 297

Step 2. Replace each number with its prime factorization.

$$\frac{48}{54} = \frac{2 \times 2 \times 2 \times 2 \times 3}{2 \times 3 \times 3 \times 3}$$

Step 3. Cancel all the factors common to both numbers. (see p.304)

$$\frac{48}{54} = \frac{\cancel{2} \times 2 \times 2 \times 2 \times \cancel{3}}{\cancel{2} \times 3 \times 3 \times \cancel{3}} \quad \text{or} \quad \frac{2}{2} \times \frac{3}{3} = \frac{2 \times 2 \times 2}{3 \times 3}$$

Or you may arrange the factors as a fraction equals 1.
The GCF of 48 & 54 is the product of their common factors: 2 x 3 = 6.

Step 4. Multiply the remaining factors in the numerator to find the new numerator, and the remaining factors in the denominator to get the new denominator.

$$\frac{48}{54} = \frac{2 \times 2 \times 2}{3 \times 3} = \frac{8}{9}$$

8 & 9 are relatively prime.
Their GCF is 1

Reducing A Fraction To Lowest Terms: Comments on the Methods

1. **Common Factors Method** (p.306) **vs. Prime Factorization Method** (p.309)
   - * Finding the GCF of two numbers by "Common Factors Method" is cumbersome and impractical when the numerator and the denominator are large numbers.
   - * Finding the GCF of two numbers by Prime factorization is simpler than "Common Factors Method."

2. **Invert Short Division Method** (p.307) **vs. Prime Factorization Method** (p.309)
   - * In regular Invert Short Division Method, you **divide both numbers by any common factor - prime or composite.** To write a fraction in lowest terms, the steps of finding the GCF and dividing the fraction by their GCF can be omitted.
   - * In Prime Factorization-Invert Short Division, you **divide each number individually by its prime factor(s) only.** To write a fraction in lowest terms, the steps of finding the GCF and divide the fraction by their GCF can not be omitted.

3. **Successive Deduction Method** (p.308) **vs. Invert Short Division Method** (p.307)
   - * Successive Deduction Method is in fact similar to Invert Short Division Method. However, it is the only method that does not require to find the GCF of the numerator and the denominator.

LCM & Changing Unlike Fractions To Like Fractions (Read "GCF vs. LCM" p.280)

Keep in mind that fractions can be **added, subtracted,** or **compared** only if they are like fractions (having the same number for the denominators). Changing unlike fractions (having different numbers for the denominators) to like fractions is a two-step process:

1st. You must know how to find the LCM of two or more numbers.
You will be able to find the LCM of two or more numbers,
a) if you know **what multiples are**, and **how to find them** (pp 283,312)
b) if you know **primes** and **prime factorization** (p.295)
c) if you know **exponents** (p.38).

2nd. You must know how to change unlike fractions to like fractions.
You will be able to change unlike fractions to like fractions,
a) if you know **"1 property of multiplication"** (p.298).
b) if you know how to write **"1" in fraction form** (p.299).
c) if you know how to write **equivalent fractions using the LCM** (p.320).

Connection: The concept of multiples and LCM is used in Algebra to solve the system of equations by elimination.

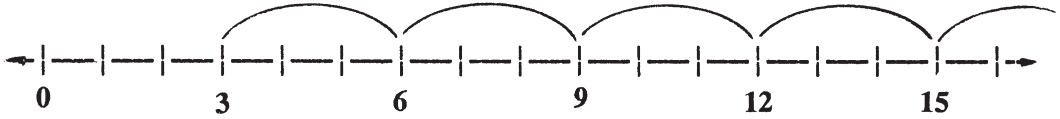
Finding Multiples: A How-To (Read first "Multiples" p.283)

How To Find the Multiples of 3:

To find the multiples of a number, **multiply the number by 0, 1, 2, 3,... the consecutive whole numbers.** The products are the multiples of the number. The following shows how to find the multiples of 3:

	3x0	3x1	3x2	3x3	3x4...	**multiply 3 by 0, 1, 2, 3,...**
Multiples of 3:	**0**	**3**	**6**	**9**	**12...**	**products are multiples of 3**

Another way to find the multiples of a number is **to count by the number (skip count).** To find the multiples of 3, count by 3s which can be shown on a number line.



Count by 3s:	**3**	**6**	**9**	**12**	**15 ...**	**multiples of 3**
	+3	+3	+3	+3		**add 3**

Observe the pattern in the sequence of the multiples of 3. The rule for the sequence is adding 3.

Finding Least Common Multiple (LCM): Method 1

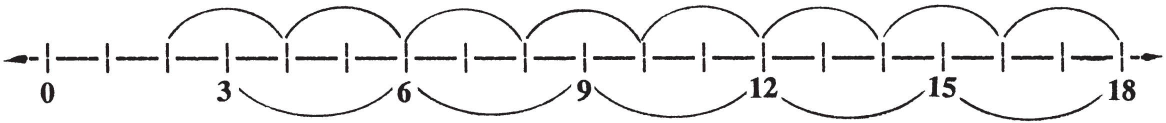
Finding the LCM of 2 and 3 by Using The Number Line:

Since the number line can be used to find the multiples of a number **(see p.312)**, it can also be used to find the least common multiples (LCM) of two numbers. Here is a how-to:

1st, Draw the number line.

2nd, Show the **multiples of 2** on the number line - **above.**

count by 2s



count by 3s

3rd, Show the **multiples of 3** on the number line - **underneath.**

4th, The **common multiples** of 2 and 3 are the numbers(s) **where the multiples of 2 and 3 meet on the number line: 6, 12, 18,...**

5th, The LCM of 2 and 3 is **the first non-zero number where the multiples of two numbers meet: 6**

Finding Least Common Multiple (LCM): Method 2 (Review first p.312)

Finding the LCM of 4 and 6 by Common Multiples Method:

Step 1. List the first few multiples of the smaller number. Then list the multiples of the larger number.

multiples of 4: 0, 4, 8, **12**, 16, 20, **24**, ...
multiples of 6: 0, 6, **12**, 18, **24**, 30, 36, ...

If no common multiple is found, list several additional multiples of the smaller number Then list multiples of the greater number until a common multiple is found. That multiple is the LCM. In that case, Step 2 & 3 are not needed.

Step 2. Identify the multiples which are **common to both numbers.**
The common multiples of 4 and 6 are 12, 24,... (the multiples of 12)

If zero is included in the list of common multiples, then the LCM will always be 0 because 0 is the first multiple of every number

Step 3. Choose the Least Common Multiples (LCM). The LCM is the **smallest non-zero number** common to both numbers.
The LCM of 3 and 4 is 12.

Finding Least Common Multiple (LCM): Method 3 (Review first P 312)

Finding the LCM of 4 & 15 by the Multiples of Larger Number:

First, list the first few multiples of the larger number of the two numbers. Then, **test each multiple to see whether it is divisible by the smaller number.** The first multiple of the larger number that can be divided by the smaller number evenly is the LCM.

The multiples of 15:	**0,**	**15,**	**30,**	**45,**	**60,...**
Is the multiple divisible by 4?:	**No**	**No**	**No**	**No**	**Yes**

So, the common multiples of 4 and 15 are 60, 120, 180,... (the multiples of 60)

The LCM of 4 and 15 is 60 or **The LCM(4, 15) = 60**

General Rules:

- * **If two numbers are relatively prime** like 4 and 15 (which means their GCF is 1), the LCM of the two number is **their product:** 4 x 15 = 60.
- * **If the larger number is a multiple of (or divisible by) the smaller number** like 4 and 32, **the LCM is the larger number of the two:** 32 (32 ÷ 4 = 8)

Finding Least Common Multiples (LCM): Method 4

Finding the LCM of 18, 48, & 60 by Invert Short Division:

Step 1. Divide the numbers (dividends) ***by any prime number*** **that will divide evenly into most of them. Bring down the number** ***which is not divisible by the prime.*** Continue the process **until the quotients are all primes.**

$$\begin{array}{r|rrr} 2 & 18 & 48 & 60 \\ \hline 2 & 9 & 24 & 30 \\ \hline 3 & 9 & 12 & 15 \\ \hline 2 & 3 & 4 & 5 \\ \hline & 3 & 2 & 5 \end{array}$$

(9 is not divisible by 2, bring 9 down.)

(3 & 5 are not divisible by 2, bring them down.)

All the divisors and quotients are prime numbers.

Step 2. To find the LCM of 18, 48, and 60, **multiply all the divisors and all the quotients.**

The LCM = $2 \times 2 \times 3 \times 2 \times 3 \times 2 \times 5 = 2^4 \times 3^2 \times 5 = 720$

Note: "Invert Short Division" (p.297) is one of the two methods used in prime factorization. The other is by **"Factor Tree".** (p.296)

Finding Least Common Multiple (LCM): Method 5

Finding the LCM of 18, 48, & 60 by Prime Factorization Method:

Step 1. Factor each number into prime by "Factor Tree Method" described on page 296. It is omitted here to save the space. (With some practice, you can factor a number into primes mentally.)

Step 2. Write each number as the product of prime factors. Then write the identical factors in **exponent.** (See p.38)

$$18 = 2 \times 3 \times 3 = 2 \times \enclose{circle}{3^2}$$
$$48 = 2 \times 2 \times 2 \times 2 \times 3 = \enclose{circle}{2^4} \times 3$$
$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times \enclose{circle}{5}$$

All the different factors are: 2, 3, 5

Step 3. To find the LCM, multiply together the highest powers of **each** of the different factors that occur in all three prime factorizations:

- the highest powers of 2 is 4 -- 2^4 (from 48)
- the highest powers of 3 is 2 -- 3^2 (from 18)
- the highest powers of 5 is 1 -- 5 (from 60)

Therefore, the LCM (18, 48, 60) = $2^4 \times 3^2 \times 5 = 720$

Finding Least Common Multiple (LCM): Comments on the Methods

1. **The Number Line Method** **(p.313)**
   This method is given to show the beginners that multiples and counting by a number (skip count) are related.

2. **The Common Multiples Method** **(p.314)**
   This method is cumbersome when the numbers are large or when it involves more than two numbers.

3. **The Multiples of Larger Number Method** **(p.315)**
   This method is simple. All you need to do is write down the multiples of the larger number and the rest can be done mentally.

4. **Prime Factorization Method** **(p.317)** **& Invert Short Division** **(p.316)**
   Prime factorization, either by factor tree or by invert short division, is an efficient method. It is the only method which can be used to find the GCF as well as the LCM as you see on page 319.

   Note: If you have mastered the **division,** know the **divisibility rules,** and remember the **prime numbers,** you can factor a number into prime *mentally* You do all the divisions in your head and write down the prime factors.

Prime Factorization: LCM vs. GCF

(Review first "Exponents" p.38)

The following shows you how to use prime factorization to find the LCM and the GCF of 36 and 40. To avoid confusion, they are placed side by side so that you may see the similarity and the difference.

Finding the LCM

Step 1.

Factor each number into primes.

36 = 2 x 2 x 3 x 3
40 = 2 x 2 x 2 x 5

Write factors in exponent form.

36 = 2^2 x 3^2
40 = 2^3 x 5

Step 2.

To find the LCM of 36 and 40, **multiply the largest exponents of each of the different factors:**

2^3 x 3^2 x 5 = 360

The LCM (36, 40) = 360

Finding the GCF

Step 1.

Factor each number into primes.

36 = 2 x 2 x 3 x 3
40 = 2 x 2 x 2 x 5

Step 2.

To find the GCF of 36 and 40, **multiply the factors that are common to both numbers:**

2 x 2 = 4

The GCF (36, 40) = 4

Changing Unlike Fractions to Like Fractions: A How-To (1)

Changing 5/18 and 7/12 to Like Fractions Using the LCM:

Step 1. Find the LCM of 18 and 12. The LCM of 18 & 12 is **36.**
The LCM becomes **the common denominator (LCD) of 5/18 & 7/12.**

Step 2. Write equivalent fractions of 5/18 and 7/12 with 36 as the denominator.
Find the number by which you multiply the denominators 18 and 12 respectively to get 36.

$$\begin{tikzcd} \frac{5}{18} \arrow[rr, bend right, "\times n"'] & = & \frac{?}{36} & & \frac{7}{12} \arrow[rr, bend right, "\times m"'] & = & \frac{?}{36} \end{tikzcd}$$

$n = 36 \div 18 = \mathbf{2}$

$m = 36 \div 12 = \mathbf{3}$

Then, Multiply 5/18 by **2/2 (= 1)** and 7/12 by **3/3 (= 1).**

$$\begin{tikzcd} \frac{5}{18} \arrow[rr, bend left, "\times 2"] \arrow[rr, bend right, "\times 2"'] & = & \frac{10}{36} & & \frac{7}{12} \arrow[rr, bend left, "\times 3"] \arrow[rr, bend right, "\times 3"'] & = & \frac{21}{36} \end{tikzcd}$$

$$\frac{5}{18} \times \frac{2}{2} = \frac{10}{36} \qquad \frac{7}{12} \times \frac{3}{3} = \frac{21}{36}$$

Changing Unlike Fractions to Like Fractions: A How-To (2) `(Review first p.303)`

Changing 5/18 and 7/12 to Like Fractions By Cross-Multiplying:

Step 1. `Same as in Method 1, see last page.`

Step 2. `Write equivalent fractions with 36 as the denominator.`
Since we know the new denominator, we can use cross multiplication to find the new numerator. Let n represent the unknown numerator.

Cross Multiplication

$$\frac{5}{18} = \frac{n}{36} \qquad (18 \text{ x } n = 5 \text{ x } 36,\ n = 10) \qquad \frac{5}{18} = \frac{10}{36}$$

$$\boxed{n = \frac{5 \text{ x } \overset{2}{\cancel{36}}}{\underset{1}{\cancel{18}}} = 10} \qquad \boxed{n = \frac{7 \text{ x } \overset{3}{\cancel{36}}}{\underset{1}{\cancel{12}}} = 21}$$

$$\frac{7}{12} = \frac{n}{36} \qquad (12 \text{ x } n = 7 \text{ x } 36,\ n = 21) \qquad \frac{7}{12} = \frac{21}{36}$$

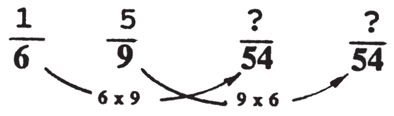
`So, the like fractions of 5/18 and 7/12 are` **10/36 and 21/36.**

Changing Unlike Fractions to Like Fractions: A How-To (3)

Changing 1/6 and 5/9 by Multiplying the Given Denominators:

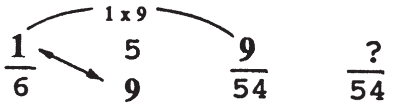
Step 1. To find the common denominator, multiply the given denominators:

6 x 9 = 54



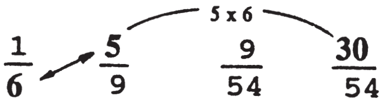
Step 2. To find the numerator of the first fraction, multiply its numerator by the other denominator

1 x 9 = 9



Step 3. To find the numerator of the second fraction, multiply its numerator by the first denominator:

5 x 6 = 30



$$\frac{1}{6} + \frac{5}{9} = \frac{1 \times 9}{6 \times 9} + \frac{5 \times 6}{9 \times 6} = \frac{(1 \times 9) + (5 \times 6)}{(9 \times 6)}$$

It looks like this using addition.

So, the like fractions of 1/6 and 5/9 are **9/54 and 30/54.**

Changing Unlike Fractions to Like Fractions: A How-To (4)

Changing 1/2, 3/4 & 7/16 to Like Fractions: Special Case

General Rule: ***If one denominator*** (the larger number of the two or the largest number of three) ***is a multiple of the other denominator(s)***, the LCD is that number. **(See also page 315)**

First, *examine* the denominators 2, 4, 16. You notice that they are multiples of 2 and 16 is a multiple of 2 and 4:

16 is a multiple of 2 because 2 x **8** = 16.
16 is a multiple of 4 because 4 x **4** = 16.
The LCD of the equivalent fractions is **16.**

Next, *write equal fractions* of 1/2 and 3/4 **with 16 as the denominator.**

Multiply 1/2 by 8/8 (= 1) = 8/16 and
Multiply 3/4 by 4/4 (= 1) = 12/16.

So, the like fractions are **8/16, 12/16, & 7/16.**

Changing Unlike Fractions to Like Fractions: A How-To (5)

Changing 1/2, 4/7 and 2/9 to Like Fractions: Special Case

General Rule: ***If the denominators*** of two or more fractions are ***relatively prime*** which means their GCF is 1, the LCD of the fractions is **the product of the denominators.**

First, *examine* the denominators 2, 7, 9. You notice that they are relatively prime because their only common factor is 1.
So, the LCD of the fractions is **2 x 7 x 9 = 126**

Next, *write equivalent fractions* with 126 as the denominator.
126 ÷ 2 = **63** so, **multiply** 1/2 by **63/63** = 63/126
126 - 7 = **18** so, **multiply** 4/7 by **18/18** = 72/126
126 - 9 = **14** so, **multiply** 2/9 by **14/14** = 28/126

Shortcut: If 2 x 7 x 9 = 126 then, 126 ÷ 2 = 7 x 9 = 63
126 -- 7 = 2 x 9 = 18
126 ÷ 9 = 2 x 7 = 14

So, the like fractions are **63/126, 72/126, 28/126.**

Changing Unlike Fractions to Like Fractions: Comments on the Methods

Review first "LCM & Changing Unlike Fractions to Like Fractions" (p.311) and "Finding LCM" (p.317) because they are all related.

1. **The LCM Method** (p.320) **& The Cross-Multiplying Method** (p.321)
   These two methods are the same in using the LCM. But they differ in step 2 as to how to write equivalent fractions. Both methods are also used in "Ratios & Proportions".

2. **Multiplying the Given Denominators Method** (p.322)
   In case you forget how to find the LCM/LCD you will still be able to change unlike fractions to like fractions if you know this method. But this method could result in computing larger numbers than it is necessary. **For example,** the common denominator of 1/6 and 5/9 is **18** by using the LCM method, and **54, 3 times larger,** by using this method.

3. **Two Special Cases** (p.323) and (p.324)
   You would be able to use these two special cases, if you know the concepts of multiples, the rules of divisibility (p.288), and have memorized some of the prime numbers (p.291).

Finding The Reciprocal of A Number: A How-To

* To Find the Reciprocal of A Whole Number 5:

$$5 \longrightarrow \frac{5}{1}$$

$$\begin{tikzcd} 5 \arrow[dr] & 1 \\ 1 \arrow[ur] & 5 \end{tikzcd}$$

1st. Write 5 as a fraction **with the denominator 1.**

2nd. To find the reciprocal, ***invert*** **the fraction –**
Change the places of the numerator and the denominator

* To Find the Reciprocal of A Mixed Number 4 3/8:

$$4\frac{3}{8} \longrightarrow \frac{35}{8}$$

$$\begin{tikzcd} 35 \arrow[dr] & 8 \\ 8 \arrow[ur] & 35 \end{tikzcd}$$

1st. Change the mixed number to **improper fraction.**

2nd. To find the reciprocal, ***invert*** **the fraction –**
Change the places of the numerator and the denominator

*** The Test of Reciprocal:** "*Two numbers are reciprocals of each other, if their product is 1.*"

$$\frac{\cancel{5}}{\cancel{1}} \times \frac{\cancel{1}}{\cancel{5}} = 1 \quad \text{and} \quad \frac{\cancel{35}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{35}} = 1$$

The factors cancel out.

Connection: You need this knowledge in doing **division of fractions.**

Reciprocal or Multiplicative Inverse: Changing Division To Multiplication

Multiplicative inverse is another name for reciprocals. The following example shows you how the reciprocal or the multiplicative inverse is used to change division to multiplication.

Example: Divide 6 ÷ 3/4

$$6 \div \frac{3}{4} \overset{①}{=} \frac{6}{\frac{3}{4}} \overset{②}{=} \frac{6 \times \frac{4}{3}}{\frac{\cancel{3}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{3}}} \overset{③}{=} \frac{6 \times \frac{4}{3}}{1} \overset{④}{=} 6 \times \frac{4}{3}$$

① Write the division as **a complex fraction.** (See p.91)

② ***Multiply both*** the numerator and the denominator of the complex fraction **by 4/3, the reciprocal of the denominator 3/4.**

③ The denominator **equals 1** (The product of 3/4 and its reciprocal 4/3 equals 1)

④ ***Therefore, To divide by a number is the same as to multiply by its reciprocal.***

Connection: In Algebra, we do not use division. We change division to multiplication **by using the property of multiplicative inverse** as seen above.

Summary

(Factors & Multiples)

* Factors are exact divisors. The greatest common factor (GCF) is the greatest factor that is common to both the numerator and the denominator of a given fraction.
* Multiples are the products of a number and any natural numbers. The least common multiple (LCM) of two or more numbers is the least multiple that is common to those numbers.
* We use division to find the factors of a given number Prime factorization is the process of writing a composite number as the product of prime factors.
* The "1 property of multiplication & division" with 1 written in fraction form allow us to write equivalent fractions.
* Cancellation is used to simplify computation by eliminating the common factors before multiplying.
* To write equivalent fraction(s) of lower terms, we divide both the numerator and the denominator of a given fraction by their common factor By using GCF, we reduce a fraction to lowest terms.
* To write equivalent fraction(s) of higher terms, we multiply both the numerator and the denominator of a given fraction by the same non-zero number By using LCM, we change unlike fractions to like fractions.
* Prime factorization method can be used to find both the GCF and the LCM.
* To write the reciprocal of a fraction is to invert the numerator and the denominator of the fraction.

Part IV. Fraction Operations

C. Addition

Table of Contents

General Procedure For Adding Fractions

To Add Like Fractions:

1st. Add the numerators. (The sum is the new numerator of the fraction.)
2nd. Place the sum (step 1) over the same common denominator.
3rd. Reduce the fraction to **lowest terms**. For example:

(a) 1/4 + 1/4 = **2/4** = **1/2**	(reduce to its lowest term)
(b) 3/7 + 4/7 = **7/7** = **1**	(change improper fraction to 1)
(c) 5/8 + 7/8 = **12/8** = 1 4/8 = **1 1/2**	(change to mixed number)

To Add Unlike Fractions:

A. If the larger denominator is a multiple of the smaller one, use the larger denominator as **the least common denominator (LCD)** of the given fractions.

- **1st. Write an equivalent fraction** for the other fraction using the LCD as the denominator.
- **2nd.** Add the like fractions by following the steps given above.

B. If *no* denominator is a multiple of the other, then,

- **1st. Find the least common denominator of the given fractions (use LCM).** (p.317)
- **2nd. Write equivalent fractions for the given fractions using the LCM as the common denominator.**
- **3rd.** Add the like fractions by following the steps given above.

General Procedure For Adding Mixed Numbers

To Add Mixed Numbers with Like Fractions:

1st. **Add the whole number parts** of the fractions.

2nd. **Add the fractional parts** by following the steps given on last page.

3rd. **Add the sum** of the whole numbers and the sum of the fractions.

To Add Mixed Numbers with Unlike Fractions:

A. **If one denominator is a multiple of the other,** use that denominator as the least common denominator.

- **1st.** **Write equivalent fraction with the same denominator for the other.**
- **2nd.** Add the like fractions by following the steps given on last page.
- **3rd.** Add the whole number parts.
- **4th.** Add the sum of the whole numbers and the sum of the fractions.

B. If ***no*** denominator is a multiple of the other, then,

- **1st.** **Find the least common denominator of the given fractions using the LCM.**
- **2nd.** **Write equivalent fractions** so all fractions have the same denominator.
- **3rd.** Add the like fractions by following the steps given on last page.
- **4th.** Add the whole number parts.
- **5th.** Add the sum of the whole numbers and the sum of fractions.

Note: The addition and subtraction of fractions can be done either horizontally or vertically; but the multiplication and division of fractions should be done horizontally

Adding Like Fractions

Example: Add 1/8 + 5/8 + 7/8

$$\frac{1}{8} + \frac{5}{8} + \frac{7}{8} = \frac{1+5+7}{8} = \frac{13}{8} = 1\frac{5}{8}$$

To add like fractions, it is much easier to do it **horizontally** (side way). Adding like fractions is a simple operation.

Procedure:

1st. Add the numerators: 1 + 5 + 7 = 13 ← **the sum**

2nd. Place the sum over the same common denominator: 13/8

3rd. Change the improper fraction to a mixed number: 1 5/8

(A mixed number is a whole number plus a fraction with "+" omitted.)

Review: To change a mixed number to a fraction

$$\begin{array}{rrl} & 1 & 5/8 \quad (\text{remainder/divisor}) \\ \text{divisor} \nearrow 8 & \overline{)\,13} & \\ & -\ 8 & \\ \hline & 5 & \leftarrow \text{remainder} \end{array}$$

Adding Mixed Numbers With Like Fractions

Example: Add 2 4/5 + 7 2/5 + 3 1/5

Horizontal Method

$$2\frac{4}{5} + 7\frac{2}{5} + 3\frac{1}{5}$$

$$= 12 + \frac{7}{5} = 12 + 1 + \frac{2}{5} = 13\frac{2}{5}$$

Vertical Method

$$\begin{array}{r} 2\ 4/5 \\ 7\ 2/5 \\ +\ 3\ 1/5 \\ \hline 12\ 7/5 \end{array} = 13\ 2/5$$

To add mixed numbers, **vertical method seems to be a better method.** **We add the whole number parts and fractional parts separately.**

Procedure:

1st. Add the whole number parts: 2 + 7 + 3 = 12 ← **the sum**

2nd. Add the fractional parts: $\frac{4 + 2 + 1}{5} = \frac{7}{5} = 1\frac{2}{5}$ ← **the sum**

3rd. Add the sum of whole numbers and the sum of fractional parts:

$$12 + \frac{7}{5} = 12 + 1 + \frac{2}{5} = 13\frac{2}{5} \quad (7/5 = 1\ 2/5 = 1 + 2/5)$$

Adding Whole Number And Like Fractions

Example: Add 7 + 1/9 + 5/9

Horizontal Method

$$7 + \frac{1}{9} + \frac{5}{9}$$

$$= \quad 7 + \frac{6}{9} \quad = \quad 7 + \frac{2}{3} \quad = \quad 7\frac{2}{3}$$

Vertical Method

$$\begin{array}{rr} & 7 \\ & 1/9 \\ + & 5/9 \\ \hline & 7\ 6/9 \end{array} = 7\ 2/3$$

Procedure:

1st. Add the like fractions: $\dfrac{1+5}{9} = \dfrac{6}{9}$

2nd. Reduce the fraction, 6/9, to lowest terms:

$$\frac{6}{9} = \frac{6 \div 3}{9 \div 3} = \frac{2}{3}$$

Divide the numerator and the denominator by their common factor. 3/3 = 1

3rd. Add the whole number and the fraction:

$$7 + \frac{2}{3} = 7\frac{2}{3}$$

Write the whole number and the fraction as *a mixed number.*

Adding Unlike Fractions

Example: Add 3/8 + 5/16 + 3/4

Horizontal Method:

$$\frac{3}{8} + \frac{5}{16} + \frac{3}{4}$$

$$= \frac{6}{16} + \frac{5}{16} + \frac{12}{16} = \frac{23}{16} = 1\frac{7}{16}$$

Vertical Method:

$$\begin{array}{rcccc}
\frac{3}{8} & = & \frac{3 \times 2}{8 \times 2} & = & \frac{6}{16} \\
\frac{5}{16} & & & = & \frac{5}{16} \\
+\ \frac{3}{4} & = & \frac{3 \times 4}{4 \times 4} & = & \frac{12}{16} \\
\hline
 & & & & \frac{23}{16} = 1\frac{7}{16}
\end{array}$$

Procedure:

1st. Find the LCM of 8, 16, and 4. The LCM (8, 16, 4) is 16
We know: *16 is a multiple of 8* (8 x 2 = 16) and
16 is also a multiple of 4 (4 x 4 = 16).

2nd. Write equivalent fractions for 3/8 and 1/4 *with 16 as the denominator.*
Multiply 3/8 **by 2/2** and 3/4 **by 4/4.** (See Vertical Method)

3rd. Add the like fractions.

4th. Change the **improper fraction** to **a mixed number.**

Adding Mixed Numbers With Unlike Fractions

Example: Add 4 2/3 + 6 1/3 + 2/9

Horizontal Method:

$$4\frac{2}{3} + 6\frac{1}{3} + \frac{2}{9}$$

$$= 4\frac{6}{9} + 6\frac{3}{9} + \frac{2}{9} = 10\frac{11}{9} = 11\frac{2}{9}$$

Vertical Method:

$$\begin{array}{rcr} 4\frac{2}{3} & = & 4\frac{6}{9} \\ 6\frac{1}{3} & = & 6\frac{3}{9} \\ + \quad \frac{2}{9} & = & \frac{2}{9} \\ \hline & & 10\frac{11}{9} = 11\frac{2}{9} \end{array}$$

Procedure:

1st. Find the LCM of 3 and 9. The LCM of 3 and 9 is 9.
We know *9 is a multiple of 3.*

2nd. Write equivalent fractions for 2/3 and 1/3 *with 9 as the denominator.*
Multiply 2/3 and 1/3 by 3/3 respectively.

3rd. Add the like fractions. The sum is 11/9.

4th. Change the improper fraction, 11/9, to a mixed number: 1 2/9

5th. Add the whole numbers and the fraction: 10 + 1 + 2/9 = 11 2/9

Part IV. Fraction Operations

D. Subtraction

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General Procedure For Subtracting Fractions (Review first pp.313-318)

Keep in mind that **fractions must have the same common denominators,** before they can be added, subtracted, or even compared. **Make sure you know how to find LCM/LCD before you do the subtraction of the unlike fractions.**

Subtract Like Fractions:
1st. **Subtract the numerators** - (**The difference** is the new numerator.)
2nd. Write the difference of the numerators over the same denominator.
3rd. Always write the answer **in lowest terms.**

Subtract Unlike Fractions:
1st. ***Find the least common denominator (LCD) of the given fractions.***
Review pages 320-325 before you add or subtract unlike fractions.
2nd. **Rewrite the given fractions as like fractions *with the LCD as the common denominator.***
3rd. Subtract the like fractions by following the steps given above.

Subtract Mixed Numbers Without Borrowing:
1st. Subtract the fractional parts.
  a) **If like fractions,** follow the subtraction of like fractions.
  b) **If unlike fractions,** follow the subtraction of unlike fractions.
2nd. Subtract the whole number parts.

General Procedure For Subtracting Mixed Numbers With Borrowing

As with whole numbers and decimals, the subtraction of mixed numbers sometimes involves borrowing. It occurs **when the fraction of the minuend** (the top number) ***is less than*** **the fraction of the subtrahend** (the bottom number). In borrowing, follow the steps given below:

1st. If **unlike fractions**, change them to **like fractions using the LCM.**
2nd. After Step 1, if the fraction of the bottom number *is larger than* the fraction of the top number, **make the fraction of the top number larger** in the following way:

- a) ***Borrow 1 from the whole number.*** (Remember to reduce the whole number by 1.)
- b) **Write the 1 as a like fraction using the same denominator.**
  Example: If the denominator of like fractions is 5, write 1 as 5/5.
  If the denominator of like fractions is 9, write 1 as 9/9.
- c) Add the 1, written in fraction, to the fractional part of the number.
  It makes the minuend **an improper fraction.**

3rd. Subtract the fractional parts.
4th. Subtract the whole number part.

Note: we can add or subtract fractions either horizontally or vertically.

Subtracting Like Fractions (Fractions With Like Denominators)

Example: Subtract 8/9 - 2/9

$$\frac{8}{9} - \frac{2}{9} = \frac{8-2}{9} = \frac{6}{9} = \frac{2}{3}$$ Divide 6/9 by 3/3

Procedure:

1st. Subtract the numerators: 8 - 2 = 6 ← the difference

2nd. Write the difference over the same common denominator: 6/9

3rd. Reduce the fraction to lowest terms: 6/9 = 2/3.

Example: 6/7 - 1/7

$$\frac{6}{7} - \frac{1}{7} = \frac{6-1}{7} = \frac{5}{7}$$

Procedure:

1st. Subtract the numerators: 6 - 1 = 5 ← the difference

2nd. Write the difference over the same common denominator: 5/7

Note: Subtraction of like fractions is as simple as the addition of like fractions.
It is much easier to do it in horizontal way

Subtracting Mixed Numbers With Unlike Fractions

Example: Subtract 6 3/4 - 2 5/7

Remember: Fractions must have **the same denominators** before they can be subtracted (or added, or compared).

$$\begin{array}{rcrcl} 6\frac{3}{4} & = & 6\frac{21}{28} & \longleftarrow & \frac{3}{4} \times \frac{7}{7} = \frac{21}{28} \\[2ex] -\ 2\frac{5}{7} & & 2\frac{20}{28} & \longleftarrow & \frac{5}{7} \times \frac{4}{4} = \frac{20}{28} \\ \hline & & 4\frac{1}{28} & & \end{array}$$

Procedure:

1st. Change the **unlike fractions to like fractions using the LCM**

Since 4 and 7 are relatively prime (See p.293),
the LCM of 4 and 7 is 28, the product of 4 and 7.

2nd. Write equivalent fractions *with 28 as the common denominator.*

3rd. Subtract the fractional part.

4th. Subtract the whole number part.

Subtracting A Mixed Numbers From A Whole Number (With Borrowing)

Example: Subtract 5 - 2 3/8

Remember: 1 can be written as a fraction with the numerator equals the denominator.

$$\begin{array}{r} 5 \\ -\;2\,\frac{3}{8} \\ \hline \end{array} \quad = \quad \begin{array}{r} 4\,\frac{8}{8} \\ -\;2\,\frac{3}{8} \\ \hline 2\,\frac{5}{8} \end{array} \longleftarrow$$

$$5 = 4 + 1 = 4 + \frac{8}{8}$$

(8/8 = 1 because any number divided by itself is 1.)

Procedure:

1st. ***Borrow 1 from 5.*** The whole number 5 becomes 4.

Write the 1 borrowed as a fraction *with 8 as the denominator:*

1 = 8/8 ← an improper fraction

2nd. Subtract the like fractions: 8/8 - 3/8 = 5/8

3rd. Subtract the whole number: 4 - 2 = 2

Note: Subtracting a fraction from a whole number follows the same procedure.

Subtracting Mixed Numbers (With Borrowing)

Example: `Subtract 3 1/4 - 1 3/4`

$$\begin{array}{rcrcl} 3\frac{1}{4} & = & 2\frac{5}{4} & \longleftarrow & 3\frac{1}{4} = (2+1)+\frac{1}{4} = 2+\frac{4}{4}+\frac{1}{4} = 2\frac{5}{4} \\ -\,1\frac{3}{4} & & 1\frac{3}{4} & & \\ \cline{1-1}\cline{3-3} & & 1\frac{2}{4} & = & 1\frac{1}{2} \end{array}$$

Procedure:

1st. **Since the fraction of the minuend (the top number)** ***is less than*** **the fraction of the subtrahend (the bottom number),**
- a) ***Borrow 1 from 3,*** **the whole number 3 becomes 2.**
- b) **Write the 1 borrowed as a fraction** ***with 4 as the denominator:*** **1 = 4/4**
- c) **Add 4/4 to 1/4 to makes it an improper fraction:** 5/4

2nd. Subtract the fractional part: 5/4 - 3/4 = 2/4

3rd. Subtract the whole number part: 2 - 1 = 1

4th. **Reduce** the fraction **to lowest terms:** 2/4 to 1/2

Summary
(Addition & Subtraction)

- Unlike fractions must be changed to like fractions with the same denominators before they can be added or subtracted.
- The order in which we add fractions does not affect the sum, but the order in which we subtract fractions does affect the difference. We always subtract smaller number from a larger number because subtraction is not commutative.
- To add like fractions, add the numerators and write the sum over the common denominator
- To subtract like fractions, subtract the smaller numerator from the larger numerator and write the difference over the same denominator
- To add or to subtract mixed numbers, add or subtract the whole number parts and fractional parts separately. Then add the two sums or differences together
- If subtraction requires borrowing, borrow 1 from the whole number part and reduce that number by 1 Write the 1 borrowed as a like fraction of the fraction you are adding to. This process makes the minuend an improper fraction which is larger than the subtrahend.
- At the end of computation, an improper fraction must be changed to a mixed numbers and the fraction must be written in lowest terms.

Part IV. Fraction Operations

E. Multiplication

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General Procedure For Multiplying Fractions

The sale signs, **"1/2 off" and "1/3 off"**, we see in stores are the examples of multiplication of fractions. **Multiplication of fractions is rather simple.**

Before multiplying, change mixed numbers to improper fractions; whole numbers to fractions. Then follow either method A or method B:

Method A. Multiply First, Then Reduce

Step 1. Multiply the numerators, then multiply the denominators.
Write the product of the numerators over the product of the denominators.
Step 2. Simplify the fraction or reduce the fraction to lowest terms.

Method B. Cancel First, Then Multiply - A shorter method

Step 1. *Cancel the common factors* before multiplying.
* Look for number(s) that can *divide evenly* into both numerator and denominator. Or
* Use **prime factorization** and cross out the common factors.

Step 2. Multiply the numbers remaining in the numerators.
Multiply the numbers remaining in the denominators.
Write the product of numerators over the product of the denominators.

Mixed Numbers vs. Multiplication Of Fractions

In working with fractions, some students got mixed up in the following operations. It is important to avoid the similar mistakes.

1. Do not mixed up mixed number with the multiplication of fractions. Example:

(a) $4\frac{5}{8}$ -- (a mixed number) -- $= 4 + \frac{5}{8}$ (with "+" sign omitted)

(b) $4 \times \frac{5}{8}$ -- (a multiplication) -- $= \frac{4 \times 5}{8} = \frac{5}{2} = 2\frac{1}{2}$

Note: 4 x 5/8 can be written as 4(5/8), but never as 4 5/8.

2. Cancellation is used only with multiplication of fractions and not with addition. Example:

(a) $\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}} \times \frac{\overset{2}{\cancel{8}}}{\underset{3}{\cancel{9}}}$

To cancel means **to divide** the numerator and the denominator **by their common factor** because division and multiplication are inverse operations.

(b) $\frac{3}{4} + \frac{8}{9}$

Division and addition ***are NOT*** inverse operations.

Multiplying Whole Numbers By Proper Fractions And Vice Versa

Example: Multiply 2/15 x 3.

Method 1. $\frac{2}{15} \times 3 = \frac{2 \times 3}{15} = \frac{6}{15} = \frac{2}{5}$ **Or** $\frac{2}{\cancel{15}_{5}} \times \overset{1}{\cancel{3}} = \frac{2}{5}$

1st. **Multiply the numerator by the whole number.** Write the product over the denominator: 6/15.

2nd. Reduce the fraction to lowest terms.

Or **1st.** **Cancel the common factor:** divide 15 & 3 by 3.

2nd. Write the result as a fraction: 2/5.

Method 2. $\frac{2}{15} \times 3 = \frac{2}{15 \div 3} = \frac{2}{5}$

1st. **Divide the denominator by the whole number.**

(It's the same as cancel the common factor.)

2nd. Write the result as a fraction: 2/5.

Remember: When one factor is a whole number, you can either multiply the numerator by the whole number or divide the denominator by the whole number, **if the denominator is a multiple of the whole number.**

Multiplying Proper Fractions By Proper Fractions

Example: `Multiply.  15/18 x 3/5`

Method 1.

$$\frac{15}{18} \times \frac{3}{5} = \frac{15 \times 3}{18 \times 5} = \frac{45}{90} = \frac{1}{2}$$

$$\boxed{\begin{array}{r|ll} 5 & 45 & 90 \\ \hline 9 & 9 & 18 \\ \hline & 1 & 2 \end{array}}$$

Multiply the numerators; multiply the denominator.
Write the results as a fraction: 45/90.
Reduce the fraction to lowest terms. (See inside the box.)

Method 2.

$$\frac{\overset{3}{\cancel{15}}}{\underset{6}{\cancel{18}}} \times \frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{5}}} = \frac{3 \times 1}{6 \times 1} = \frac{1}{2}$$

Cancellation using slash.

Slash 15 and 5, divide each number by 5.
Slash 18 and 3, divide each number by 3.
Again divide 3 and 6 by 3, reduce to lowest terms.

Method 3.

$$\frac{15}{18} \times \frac{3}{5} = \frac{(\cancel{3} \times \cancel{5}) \times \cancel{3}}{(\cancel{3} \times \cancel{3} \times 2) \times \cancel{5}} = \frac{1}{2}$$

Cancellation using prime factoring.

Prime factor each number, **then cancel** the common factors.

Multiplying Mixed Numbers By Mixed Numbers

Example: Multiply 3 3/4 x 2 1/2

$$3\frac{3}{4} \times 2\frac{1}{2} = \frac{(3 \times 4) + 3}{4} \times \frac{(2 \times 2) + 1}{2} = \frac{15}{4} \times \frac{5}{2}$$

1st. Change mixed numbers to improper fractions **(see p.95)**

$$\frac{15}{4} \times \frac{5}{2} = \frac{75}{8} = 9\frac{3}{8}$$

$$\boxed{75 \div 8 = 9\frac{3}{8}}$$

2nd. Multiply the numerators. Multiply the denominator. write the result as a fraction: 75/8
(We *can not* use cancellation because there is no common factor.)

3rd. Change the improper fraction to a mixed number. Divide the numerator by the denominator and write the remainder as a fraction: dividend over divisor.

Multiply Mixed Numbers By Proper Fractions

Change the mixed number to improper fraction first. Then follow the same procedure.

Multiplying Whole Numbers By Mixed Numbers And Vice Versa

Example: Multiply 5 x 6 1/5

$$5 \times 6\frac{1}{5} = 5 \times \frac{(6 \times 5) + 1}{5} = \cancel{5} \times \frac{31}{\cancel{5}} = 31$$

Method 1. Use Improper Fraction

1st. Change the mixed number to an improper fraction (See p 95)

2nd. Cancel the common factor. 5 & 5 cancel out.

$$5 \times 6\frac{1}{5} = 5 \times \left(6 + \frac{1}{5}\right) = (5 \times 6) + \left(\cancel{5} \times \frac{1}{\cancel{5}}\right) = 30 + 1 = 31$$

Method 2. Use Distributive Property Over Addition

1st. Write the mixed number as the sum of a whole number and a fraction.

2nd. Multiply each addend (6 & 1/5) by 5 respectively.

3rd. Add the products obtained in step 2.

Note: The upper graders should be familiar with the second method. The distributive property is very useful.

The Word "Of" In Fractions Problems (See also "Percents" p.393)

In fraction (also percent) problems, the word **"of" means "times"**.

Example: 1/2 **of** 4/5 means 1/2 **x** 4/5 **"of" means multiplication.**

In general, there are **three basic types of fraction problems.** The following shows how to translate each into mathematic sentence:

Type 1. **2/3 of 4/7 is what number?** **To find the answer:**

2/3 x 4/7 = ? Simply multiply 2/3 by 4/7.

Type 2. **3/5 of what number is 2 1/3?**

3/5 x ? = 2 1/3 Divide 2 1/3 by 3/5.

Type 3. **What number of 2 3/8 is 2/5?**

? x 2 3/8 = 2/5 Divide 2/5 by 2 3/8.

Remember: Multiplication and division are inverse operations (See p.193)

The Word "Of" In Measurements

The word "of" is used very often in measurement involving fractions. The following units of the measurements are often omitted in word problems, because they are common knowledge and you are expected to know.

* A quarter of a day = $\frac{1}{4}$ x 24 hr = 6 hr (a day = 24 hours)
* A half of a pound = $\frac{1}{2}$ x 16 oz = 8 oz (1 pound = 16 ounces)
* Two third of a yard = $\frac{2}{3}$ x 36 in = 24 in (1 yard = 36 inches)
* A quarter of an hour = $\frac{1}{4}$ x 60 min = 15 min (1 hour = 60 minutes)
* One quarter of a dollar = $\frac{1}{4}$ x 100¢ = 25¢ (1 dollar = 100 cents)
* Three fourth of a year = $\frac{3}{4}$ x 12 = 9 months (1 year = 12 months)
* A quarter of a gallon = $\frac{1}{4}$ x 4 gts = 1 gt (1 gallon = 4 quarts)
* One third dozen = $\frac{1}{3}$ x 12 = 4 (1 dozen = 12)

Think! Can we write a dollar and thirty cents as 130 cents? Can we write 1 hour 30 minutes as 130 minutes? If yes, why yes? If not, why not?

Summary

(Multiplication)

* The multiplication of fractions is the simplest computation of the four operations of fractions.
* In adding and subtracting fractions, the mixed numbers are not changed to improper fractions. But in multiplication, mixed numbers must be changed to improper fractions before they can be multiplied.
* To multiply fractions, multiply the numerators, then multiply the denominators. Write the product of the numerators over the product of the denominators. Then, reduce the fraction to lowest terms or change to a mixed number
* A simpler way of multiplying fractions is to cancel first the factors which are common to both the numerator and the denominator, and then multiply the simplified fractions.
* The word "of" means "time" or "multiply" when it is used in word problems or measurement involving fractions. And the word "is" means "equal to."

Part IV. Fraction Operations

F. Division

Table of Content

General Procedure For Dividing Fractions

Multiplication of fraction is one of the simplest of the four arithmetic operations. The division of fractions is as simple as the multiplication of fractions.

Step 1. **Change any mixed number to an improper fraction.**
Change any whole number to a fraction.
(Write the whole number as a fraction is helpful, when you need to invert the whole number divisor.)

Step 2. ***Invert*** **the divisor,** the number **comes after** the division sign.
(To divide by a number is the same as to multiply by its reciprocal. A reciprocal is an inverted number as demonstrated on the next page.)

Step 3. **Multiply the fractions** by following the steps given on page 352

Remember:

* ***The order*** in which we **multiply** or **add** fractions ***does not*** affect the product or the sum because **both operations are commutative.**
* ***The order*** in which we **divide** or **subtract** fractions ***does*** affect the quotient or the difference because **neither is commutative.**

To divide by a number is the same as to multiply by its reciprocal. - a reciprocal is an inverted number. Here is the reason why:

Example: Divide 5 ÷ 3/4

ⓐ ⓑ ⓒ ⓓ

$$\frac{5}{\frac{\cancel{3}}{\cancel{4}}} \times \frac{\frac{4}{3}}{\frac{\cancel{4}}{\cancel{3}}} = \frac{5 \times \frac{4}{3}}{1} = 5 \times \frac{4}{3}$$

ⓐ Write the division as a **complex fraction** (p.91)

ⓑ Multiply **both** numerator and denominator by 4/3, the reciprocal of 3/4, so (4/3)/(4/3) = 1

ⓒ **The denominator becomes 1.**

ⓓ Thus, 5 ÷ 3/4 = 5 x 4/3

Example: Divide 7/9 ÷ 2/5

$$\frac{7}{9} \div \frac{2}{5} = n$$

Let's use n to stand for the **unknown quotient.**

$$\frac{7}{9} = n \times \frac{2}{5}$$

Division and multiplication are **inverse operation. (p.193)**

$$\frac{7}{9} \times \frac{5}{2} = n \times \cancel{\frac{2}{5}} \times \cancel{\frac{5}{2}}$$

Multiply **each side** of the equal sign by 5/2, **the reciprocal of 2/5. (p.326)**

$$\frac{7}{9} \times \frac{5}{2} = n$$

Thus, we change the division of fraction to a multiplication.

Dividing A Fraction By A Whole Number

Example: Divide 4/5 by 2.

$$\frac{4}{5} \div \frac{2}{1} = ?$$

Write the whole number in fraction form with 1 as the denominator.

$$\frac{\overset{2}{\cancel{4}}}{5} \times \frac{1}{\underset{1}{\cancel{2}}} = \frac{2}{5}$$

Invert the divisor 2/1 and multiply. **Cancel factors before multiplying.**

Dividing A Whole Number By A Fraction

Example: Divide 2 by 4/5.

$$\frac{2}{1} \div \frac{4}{5} = ?$$

It is understood that 2 = 2/1.

$$\frac{\overset{1}{\cancel{2}}}{1} \times \frac{5}{\underset{2}{\cancel{4}}} = \frac{5}{2} = 2\frac{1}{2}$$

Invert the divisor 4/5 and multiply. **Cancel factors before multiplying.**

Note: When you ***change the order*** of the numbers, you get the different answers because **division is not commutative.**

Dividing A Fraction By A Fraction

Example: Divide 3/7 ÷ 6/9

$$\frac{3}{7} \div \frac{6}{9} = n$$

Let's *n* stand for the unknown quotient.

$$\frac{{}^{1}\cancel{3}}{7} \times \frac{9}{\underset{2}{\cancel{6}}} = \frac{9}{14}$$

Invert the divisor 6/9 and multiply.
Cancel the factors before multiplying.

Dividing A Fraction By A Mixed Number

Example: Divide 7/9 ÷ 2 1/3

$$\begin{matrix}7\\9\end{matrix} \div \begin{matrix}7\\3\end{matrix}$$

The mixed number must be changed to improper fraction first.

$$\begin{matrix}{}^{1}\cancel{7}\\\underset{3}{\cancel{9}}\end{matrix} \times \begin{matrix}\cancel{3}^{1}\\\underset{1}{\cancel{7}}\end{matrix} = \begin{matrix}1\\3\end{matrix}$$

Invert the divisor 7/3 and multiply.
Cancel the factors before multiplying.

Remember: **All mixed numbers must be changed to improper fractions before multiplying or dividing.**

Summary

(Division)

- The order in which we multiply fractions does not affect the product. But the order in which we divide fractions does affect the quotient because division is not commutative.
- Division of fractions, like that of multiplication, requires that mixed numbers must be first changed to improper fractions.
- To divide fractions, invert the divisor then multiply The divisor is the number that comes after the division sign. To invert means to turn the fractions upside down. The denominators become the numerators and the numerators become the denominators.
- To avoid error, write the whole number divisor as a fraction with the number as the numerator and 1 as the denominator Then invert the fraction.